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LOW DIMENSIONAL ACTIONS OF SEMISIMPLE GROUPS

BY

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ABSTRACT

For a simple non-compact Lie group G with finite center we determine the smallest integer $n(G)$ such that G has an almost effective action on a compact manifold of dimension $n(G)$ and characterize the compact manifolds of dimension $n(G)$ on which G acts. We study actions of a semisimple group G on compact manifolds of dimension $n(G) + 1$ and determine the orbit structure of the action of G and its maximal compact subgroup. We give several examples to illustrate the results.

Introduction

Our goal in this paper is to obtain a fairly thorough understanding of the orbit structure of low dimensional actions of semisimple Lie groups. One of the starting points for this investigation is a classical result of Mostert [10] which asserts that if K is a compact group acting on a closed manifold M with an orbit of codimension one then the orbit space M/K is homeomorphic to S^1 or to the closed unit interval I, and in the former case, the K-orbits fiber M over S^1. The principle result of this paper is an analogue of Mostert's theorem for actions of non-compact semisimple Lie groups on closed manifolds.

One sees quickly that the direct analogue of Mostert's theorem is false for noncompact groups. For example, let $G = \mathrm{SL}(2,\mathbb{R})$ and consider the natural embedding of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(3,\mathbb{R})$ as the subgroup of matrices of the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Let P be the stabilizer in $\mathrm{SL}(3,\mathbb{R})$ of the z-axis in $\mathbb{R}^3$, and let G act on the manifold $\mathrm{SL}(3,\mathbb{R})/P = \mathbb{R}P^2$ by left translation. There are three G orbits: a fixed point corresponding to the z-axis; an $\mathbb{R}P^1$ corresponding to the $x - y$ plane; and the complement of these two compact orbits. Thus the orbit space $G\backslash\mathbb{R}P^2$ consists of three points, two of which are closed in the quotient topology. Another example is obtained by taking the adjoint representation of $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{Ad}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$, followed by the natural embedding of $\mathrm{SL}(3,\mathbb{R})$ in $\mathrm{SL}(4,\mathbb{R})$. One obtains an action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}P^3$ with orbits of dimensions 0, 1, and 2. The orbit structure in this example is already quite complicated. These examples indicate not only that Mostert's theorem is false for noncompact groups, but also that the hypothesis of an orbit of codimension one does not impose very strong restrictions on the orbit structure of an action of a non-compact group.

In order to find a hypothesis which is suitable for noncompact groups we recast a special case of Mostert's theorem. For any connected Lie group G let $n(G)$ be the minimum dimension of a closed manifold on which G acts smoothly and almost effectively. (Recall that a group action is **almost effective** if the kernel of the action is discrete.)

COROLLARY TO MOSTERT'S THEOREM: *Let K be a compact group. If K acts almost effectively on a closed manifold of dimension $n(K) + 1$, then M/K is homeomorphic to S^1, I, or a single point.*

Proof: Since the nontrivial orbits of K must have dimension at least $n(K)$, we see that either K acts transitively on M, in which case M/K is a point, or there is an orbit of codimension one , in which case Mostert's theorem applies. ∎

We prove the following analogue of Mostert's theorem.

THEOREM 6.9: *Let G be a connected semisimple real Lie group with finite center. Suppose G acts smoothly and almost effectively on a closed manifold M of dimension $n(G)+1$. Let K be the maximal compact subgroup of G. Then M/K is homeomorphic to S^1, I, or a point. In the first case the K-orbits fiber M over S^1.*

Since the G-orbits are connected K-saturated subsets of M we deduce immediately

THEOREM 6.11: *Let G be a connected semisimple non-compact Lie group with finite center acting smoothly and almost effectively on a closed manifold M of* *dimension $n(G)+1$. Let M/G be the orbit space of the action with the quotient topology. Then M/G is obtained from S^1 or I by identifying (possibly infinitely many) open intervals to points.*

As applications of this theorem we determine all closed manifolds in dimension two which admit an effective action of a noncompact simple Lie group, and all closed manifolds of dimension three which admit an effective action of a noncompact simple Lie group not locally isomorphic to $\mathrm{SL}(2,\mathbf{R})$.

To prove Theorem 6.9, it suffices (if the action is not transitive) to produce a single K-orbit of codimension one. Our approach to this is to identify the minimal sets of the G-action and then to relate the action of K to these minimal sets. A key step in this program is

COROLLARY 1.12: *Let G be a real algebraic group, and suppose G acts smoothly and almost effectively on a compact manifold M. Let $x \in M$ be a point contained in a minimal set of the action. Let G_x be the stabilizer of x, and G_x^0 its identity component. Then $N_G(G_x^0)$ is a cocompact real algebraic subgroup of G.*

We deduce from this

THEOREM 6.6: *Let G be a noncompact simple Lie group with finite center. Then $n(G)$ is the minimum codimension of a maximal parabolic subgroup of G. If G acts effectively on a compact manifold M of dimension $n(G)$ then M is a finite equivariant cover of G/S for some maximal parabolic $S \subset G$.*

In section 4, we compute, for each simple Lie group G, the minimum codimension of a maximal parabolic subgroup of G. Although this computation is of independent interest, it is also a necessary ingredient in the proof of Theorem 6.9. Note that Theorem 6.6 has an extension to semisimple Lie groups; the precise statement is given below.

This paper was motivated in part by (and employs some of the techniques used in the proof of) the following result of Zimmer:

THEOREM [21]: *Let G be a semisimple Lie group with finite center and suppose that G acts smoothly on a compact manifold M preserving an H-structure on M, where H is a real algebraic group. Then $\mathbf{R}-\mathrm{rank}(G) \le \mathbf{R}-\mathrm{rank}(H)$.*

In the special case of $H = GL(n,\mathbf{R})$, $n = \dim(M)$, the theorem just says that $\dim(M) \ge \mathbf{R}\text{-rank}(G)$. Although this special case hardly does justice to the theorem, it is interesting to note that in the special case the theorem is far from being optimal. For example, $G = \mathrm{SU}(2,2)$ has real rank 2 but does not act on a compact manifold of dimension less than 4. For $G = \mathrm{SO}(p, p+1)$, the real rank of G is p, but G does not act in dimension less than $2p-1$. On the other hand, for $G = \mathrm{SL}(n, \mathbb{R})$, $\mathbb{R}$-rank$(G) = n-1$ and G acts effectively on a compact $n-1$ dimensional manifold. These examples show that something other than the $\mathbb{R}$-rank of G controls the minimal dimension of an action, and indeed the results of this paper indicate that low dimensional actions of noncompact simple groups are in some sense controlled not by the $\mathbb{R}$-rank, but by the maximal compact subgroup. This in turn suggests a new approach to the general case of the theorem of Zimmer, in which the action is supposed to preserve a geometric structure (with algebraic structure group). We postpone this more general consideration to a subsequent paper. Note, however, that from our results it follows that for lowest-dimensional actions the isotropy groups are not unimodular, and this suggests that in low dimensions an invariant geometric structure cannot be unimodular. This fact was already observed by Zimmer.

THEOREM [20]: *Let G be a connected semisimple Lie group with finite center and no compact factors, and suppose that G acts on a compact manifold of dimension n preserving an H-structure, where H is an algebraic subgroup of $SL(n, \mathbb{R})$. Then there is an embedding of Lie algebras $\mathfrak{h} \to \mathfrak{g}$, and the representation $\mathfrak{h} \to \mathfrak{g} \to \mathfrak{sl}$ contains $\mathrm{ad}_{\mathfrak{h}}$ as a direct summand.*

In the special case of $H = \mathrm{SL}(n, \mathbb{R})$, the theorem says that $n \geq \dim(H)$. Indeed, a key step in the proof of the theorem shows that the stabilizer of almost every point is discrete, and therefore there are orbits of dimension equal to $\dim(H)$.

Finally, we note that specific cases of low dimensional actions of noncompact simple groups have been treated by several authors. For example, in [14], Schneider classified real analytic actions of $\mathrm{SL}(2, \mathbb{R})$ on surfaces. In [2] and [3], Asoh classified smooth actions of $\mathrm{SL}(2, \mathbb{C})$ on S^3 up to equivariant homeomorphism. In [16],[17], and [18], Uchida classified real analytic actions of $\mathrm{SL}(n, \mathbb{R})$ on S^m, $5 \leq n \leq m \leq 2n-2$, and real analytic actions of $\mathrm{SL}(n, \mathbb{C})$ on S^m, for $14 \leq n \leq m \leq 4n-2$.

This paper is organized as follows: In section one we describe a mapping from a G-manifold to the Lie algebra of G which we call the Gauss map. This allows us to characterize stabilizers of points in minimal sets. Sections two through four are devoted to a discussion of parabolic subgroups. In particular, we compute, for each simple real Lie algebra, the minimum codimension of a parabolic subgroup. In section five we establish some elementary facts about low dimensional real representations of simple groups. Section six contains the proofs of the main theorems. In section seven we collect some corollaries, remarks, and examples pertaining to the main theorems. In section eight we have tabulated some information from sections four and five.

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1. The Gauss map of an action

Let G be a connected Lie group acting on a topological space M. For $k \in \{1, ..., \dim(G)\}$, let $\mathrm{Gr}_k(\mathfrak{g})$ be the Grassmann manifold of k-planes in $\mathfrak{g}$. If V is a k-dimensional subspace of $\mathfrak{g}$ we denote by $[V]$ the corresponding element of $\mathrm{Gr}_k(\mathfrak{g})$. Let $\mathrm{Gr}(\mathfrak{g}) = \bigcup_{k=1}^{\dim(\mathfrak{g})} \mathrm{Gr}_k(\mathfrak{g})$, topologized as a disjoint union. If H is a subgroup of G we denote by H^0 the identity component of H. For $x \in M$, we denote by G_x the stabilizer of x in G. Define a map $\phi : M \to \mathrm{Gr}(\mathfrak{g})$ by $\phi(m) = [\mathcal{L}(G_x)]$, where $\mathcal{L}(\cdot)$ denotes "Lie algebra of".

Definition 1.1: The map $\phi : M \to Gr(\mathfrak{g})$ is called the **Gauss map** of the G action on M. ▮

Note that ϕ is a G equivariant map, where G acts on $\mathrm{Gr}(\mathfrak{g})$ via the adjoint action of G on $\mathfrak{g}$. The map ϕ is not in general continuous. The object of this section is to find appropriate subsets of M on which ϕ is continuous, and then to use ϕ to relate the structure of these sets as G-spaces to corresponding subsets of $\mathrm{Gr}(\mathfrak{g})$. The results of this section are closely related to those of [21].

Definition 1.2: A closed, G-invariant subset $L \subset M$ is a **minimal set** if L contains no proper, closed, invariant subset. ▮

If M is compact then minimal sets exist; in fact, the closure of every orbit contains a minimal set.

Definition 1.3: A subset $X \subset M$ is **locally closed** if it is open in its closure. The action of G on M is said to be **tame** if every orbit is locally closed. ▮

LEMMA 1.4: *Suppose G acts tamely on M. Then every minimal set of the action is a compact orbit.*

Proof: Let M_0 be a minimal set. If M_0 contains a noncompact orbit $G \cdot x$, then $\overline{G \cdot x} - G \cdot x$ is closed and nonempty, which contradicts the minimality of M_0. ∎

LEMMA 1.5: *Suppose G acts on two compact Hausdorff topological spaces M_1 and M_2, and $\phi : M_1 \to M_2$ is a continuous G-map. If C is a minimal set of M_1, then $\phi(C)$ is a minimal set of M_2.*

Proof: $\phi(C)$ is a closed invariant set in M_2. Let C' be a minimal set in $\phi(C)$. Then $\phi^{-1}(C') \cap C$ is a closed invariant subset of C, so by minimality, $\phi^{-1}(C') \supset C$. Thus $\phi(C) = C'$. ∎

COROLLARY 1.6: *Let $\phi : M_1 \to M_2$ be a continuous G-map and suppose G acts tamely on M_2. If C is a minimal set of M_1, then $\phi(C)$ is a compact G-orbit.*

Henceforth we assume that M is a smooth compact manifold and that G acts smoothly on M. Define a function $\psi : M \to \mathbb{N}$ by $\psi(m) = \dim(G_m)$. It is well-known that ψ is an upper semi-continuous function, i.e., that $\psi^{-1}(\{0, 1, \ldots, d\})$ is open for each $d \in \mathbb{N}$.

LEMMA 1.7: *Let $\phi : M \to \mathrm{Gr}(\mathfrak{g})$ be the Gauss map. Then $\phi|\psi^{-1}(d)$ is continuous for all $d \in \mathbb{N}$.*

Proof: Fix a positive definite quadratic form on $\mathfrak{g}$. For $X \in \mathfrak{g}$, let X^* be the vector field on M generated by X. Let $x \in \psi^{-1}(d)$, and $\{x_i\} \subset \psi^{-1}(d)$ a sequence converging to x. Suppose $\{\phi(x_i)\}$ does not converge to $\phi(x)$ in $\mathrm{Gr}_d(\mathfrak{g})$. Then we may choose a subsequence $\{x_{i_k}\}$ and unit vectors $X_k \in \mathcal{L}(G_{x_{i_k}})$ such that X_k converges to X in $\mathfrak{g}$, and $X \notin \mathcal{L}(H_x)$. But then $(X_k^*)_{x_{i_k}} = 0$, so $X_x^* = 0$, which is a contradiction. ∎

LEMMA 1.8: *If C is a minimal set of the G-action then every orbit in C has the same dimension.*

Proof: Let k be the minimum dimension of an orbit in C. Then

$$(M - \psi^{-1}(\dim(G) - k)) \cap C$$

is a closed invariant subset of C, hence equal to C by minimality. ∎

The following theorem gives crucial information about the G-action on $\mathrm{Gr}(\mathfrak{g})$ when G is a real algebraic group.

THEOREM 1.9 [4]: *Let k be a local field of characteristic 0 and G an algebraic group defined over k. Suppose G acts k-algebraically on a k-variety V. Then the action of G_k on V_k is tame (in the Hausdorff topology on V_k).*

Definition 1.10: A **real algebraic group** is a subgroup of finite index in the set of real points of a (complex) linear algebraic group defined over $\mathbb{R}$. ∎

COROLLARY 1.11: *Let G be a real algebraic group. Then G acts tamely on $\mathrm{Gr}(\mathfrak{g})$.*

Proof: The Grassmannian $\mathrm{Gr}(\mathfrak{g})$ is the set of real points of the complex Grassmann manifold $\mathrm{Gr}(\mathfrak{g}^{\mathbf{C}})$. $G^{\mathbf{C}}$ acts $\mathbb{R}$-algebraically on $\mathrm{Gr}(\mathfrak{g}^{\mathbf{C}})$, and the corollary follows from the theorem of Borel-Serre. ∎

COROLLARY 1.12: *Let G be a real algebraic group. Suppose G acts smoothly on a smooth closed manifold M. If $x \in M$ is contained in a minimal set for the G-action, then $N_G(\mathcal{L}(G_x))$ is a cocompact subgroup of G. ($N_G(\mathcal{L}(G_x))$ denotes the normalizer in G of $\mathcal{L}(G_x)$.)*

Proof: Let C be the minimal set containing x. Then by Lemmas 1.7 and 1.8, $\phi|C$ is continuous. By Corollaries 1.6 and 1.11, $\phi(C)$ is a compact orbit in $\mathrm{Gr}(\mathfrak{g})$. The orbit of $\phi(x)$ is

$$G \cdot [\mathcal{L}(G_x)] = \mathrm{Ad}(G) \cdot \mathcal{L}(G_x) = \mathrm{Ad}(G)/H$$

where H is the stabilizer of $\mathcal{L}(G_x)$ in $\mathrm{Ad}(G)$. H is cocompact in $\mathrm{Ad}(G)$, so $\mathrm{Ad}^{-1}(H)$ is cocompact in G. Clearly, $\mathrm{Ad}^{-1}(H) = N_G(\mathcal{L}(G_x))$. ∎

Note that since $N_G(\mathcal{L}(G_x))$ is a real algebraic group it has finitely many connected components, so in fact $N_G^0(\mathcal{L}(G_x))$ is cocompact in G.

COROLLARY 1.13: *Let G be a semisimple real Lie group. Suppose G acts smoothly on a smooth closed manifold M. If $x \in M$ is contained in a minimal set for the G-action, then $N_G(\mathcal{L}(G_x))$ is a cocompact subgroup of G.*

Proof: As in the proof of the preceding corollary, it suffices to see that the action of G on $\mathrm{Gr}(\mathfrak{g})$ is tame. This follows since $\mathrm{Ad}(G)$ is a real algebraic group and the action of $\mathrm{Ad}(G)$ on $\mathrm{Gr}(\mathfrak{g})$ has (up to finite cover of order at most $|\mathrm{Out}(G)|$) the same orbits as the action of G. ∎

COROLLARY 1.14: *Let G be a simply-connected, solvable real algebraic group, and suppose G acts smoothly on a compact manifold M. Then the identity component of the stabilizer of any point in a minimal set is normal in G.*

Proof: Let $H = N_G(\mathcal{L}(G_x))$, where $x \in M$ lies in a minimal set. Then G/H^0 is a compact simply connected manifold. But G and H^0 are both contractible, so G/H^0 is contractible and therefore equal to a single point. ∎

Example 1.15: The group $G = \{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \mid a \in \mathbb{R}^+,\ b \in \mathbb{R}\}$ acts naturally on the space $\mathbb{R}P^1$ of lines in $\mathbb{R}^2$. There are two orbits for this action, a fixed point corresponding to the line through $(1,0)$, and the complement of this fixed point. The stabilizer of a point in the complement is a one-dimensional non-normal subgroup of G. This action is equivalent to the adjoint action of G on $\mathrm{Gr}_1(\mathfrak{g})$. This example shows that for a general real algebraic group, an action in minimum dimension need not be transitive. ∎

LEMMA 1.16: *Let G be a Lie group acting on a compact manifold M. If the action is not transitive then any orbit contained in a minimal set has codimension at least one.*

Proof: If an orbit $G \cdot x$ has codimension 0 then it is open in M. If the action is not transitive, then $\overline{G \cdot x} - G \cdot x$ is a nonempty closed invariant set. Thus $G \cdot x$ is not contained in a minimal set. ∎

2. Cocompact algebraic subgroups

In this section we recall some standard properties of parabolic subgroups of semisimple Lie groups and establish the notation we will use in the sequel. We then use a theorem of Witte to characterize cocompact algebraic subgroups of semisimple Lie groups.

Let G be a connected semisimple real Lie group with finite center. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Then $\mathfrak{a}$ is ad-semisimple, i.e., the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ can be diagonalized. Thus we can write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda,$$

where for any linear functional $\lambda : \mathfrak{a} \to \mathbb{R}$,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | [H, X] = \lambda(H)X,\ \forall H \in \mathfrak{a}\},$$

and $\Sigma = \{\lambda \in \mathfrak{a}' \backslash \{0\} | \mathfrak{g}_\lambda \neq 0\}$. The subalgebra $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and contains a Cartan subalgebra of $\mathfrak{g}$. The set $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ is the set of **restricted roots** of $(\mathfrak{g}, \mathfrak{a})$. An element $H \in \mathfrak{a}$ is **regular** if $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$. A regular element $H \in \mathfrak{a}$ defines a subset

$$\Sigma^+ = \{\lambda \in \Sigma | \lambda(H) > 0\}$$

of **positive roots** in Σ. If Σ^+ is a set of positive roots then the set

$$\{H \in \mathfrak{a} | \lambda(H) > 0, \ \forall \lambda \in \Sigma^+\}$$

is a **Weyl chamber** of $(\mathfrak{g}, \mathfrak{a})$. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}$. The **Weyl group** $W(\mathfrak{g}, \mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ acts simply transitively on the Weyl chambers of $(\mathfrak{g}, \mathfrak{a})$. Fix a regular element $H_0 \in \mathfrak{a}$ and let Σ^+ be the associated set of positive roots. Let

$$\Pi = \{\lambda \in \Sigma^+ | \lambda \neq \alpha + \beta \text{ for } \alpha, \beta \in \Sigma^+\}.$$

Π is called a **simple system of roots**. An element $H \in \mathfrak{a}$ is **non-negative** if it is in the closure of the Weyl chamber associated to Σ^+, i.e., if $\lambda(H) \geq 0$ for all $\lambda \in \Sigma^+$. For $H \in \mathfrak{a}$, let

$$\mathfrak{s}(H) = \{X \in \mathfrak{g} | (\operatorname{ad} H - \lambda)^k X = 0, \text{for some } \lambda \geq 0, k \in \mathbb{N}\},$$

and let $S(H) = N_G(\mathfrak{s}(H))$.

Definition 2.1: A subgroup of G conjugate to a group of the form $S(H)$ is called a parabolic subgroup of G. A Lie subalgebra of $\mathfrak{g}$ conjugate to a subalgebra of the form $\mathfrak{s}(H)$ is called a parabolic subalgebra of $\mathfrak{g}$. ∎

Since the Weyl group acts transitively on the Weyl chambers we may always take H to be non-negative. There are natural decompositions

$$S(H) = Z_G(H) \ltimes S_+(H)$$
$$\mathfrak{s}(H) = Z_{\mathfrak{g}}(H) \ltimes \mathfrak{s}_+(H)$$

where

$$\mathfrak{s}_+(H) = \{X \in \mathfrak{g} | (\operatorname{ad} X - \lambda)^k H = 0, \text{for some } \lambda > 0, k \in \mathbb{N}\}$$

and $S_+(H)$ is the connected subgroup of G with Lie algebra $\mathfrak{s}_+(H)$. Let Σ_H be the roots in Σ which vanish on H. We have

$$Z_{\mathfrak{g}}(H) = \mathfrak{g}_0 \oplus \sum_{\lambda\in\Sigma_H} \mathfrak{g}_\lambda$$
$$\mathfrak{s}_+(H) = \sum_{\lambda\in\Sigma^+\backslash\Sigma_H} \mathfrak{g}_\lambda.$$

Let $\mathfrak{a}(H) = \bigcap_{\lambda\in\Sigma_H}\ker(\lambda)$, and let $\mathfrak{a}(H)^\perp$ be the orthogonal complement of $\mathfrak{a}(H)$ in $\mathfrak{a}$ with respect to the Killing form on $\mathfrak{g}$. Let $\mathfrak{m}(H)$ be the orthogonal complement of $\mathfrak{a}(H)$ in $Z_{\mathfrak{g}}(H)$. There are further decompositions

$$Z_{\mathfrak{g}}(H) = \mathfrak{m}(H) \oplus \mathfrak{a}(H)$$
$$\mathfrak{m}(H) = \mathfrak{a}(H)^\perp \oplus Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \sum_{\lambda\in\Sigma_H} \mathfrak{g}_\lambda.$$

The subalgebras $\mathfrak{m}(H)$ and $\mathfrak{a}(H)$ commute. Note that for $\lambda \in \Sigma_H$, $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \subset \mathfrak{a}(H)^\perp \oplus Z_{\mathfrak{k}}(\mathfrak{a})$. Set $\mathfrak{n}(H) = \mathfrak{s}_+(H)$. The decomposition $\mathfrak{s}(H) = (\mathfrak{m}(H) \oplus \mathfrak{a}(H)) \ltimes \mathfrak{n}(H)$ is called a **Langlands decomposition** of $s(H)$. Note that this decomposition is canonical given the choices $\mathfrak{a}$, Σ^+, and H. There is an associated decomposition

$$S(H) = (M(H)\cdot A(H)) \ltimes N(H),$$

where $M^0(H)$, $A(H)$, and $N(H)$ are the connected subgroups of G with Lie algebras $\mathfrak{m}(H)$,$\mathfrak{a}(H)$, and $\mathfrak{n}(H)$ respectively, and $M(H) = Z_K(H)M^0$. For $H \in \mathfrak{a}$ non-negative define $\Pi_H = \Sigma_H \cap \Pi$. The following well-known lemma is now more or less obvious.

Lemma 2.2: *There is a one-to-one correspondence between conjugacy classes of parabolic subgroups and subsets $\Delta \subset \Pi$. The correspondence is given by*

$$S(H) \longleftrightarrow \Pi_H,$$

where H is a non-negative element of $\mathfrak{a}$. If $\Delta \subset \Pi$ we write $S(\Delta) = S(H)$, for any $H \in \mathfrak{a}$ such that $\Delta = \Pi_H$. (We may take any $H \in \bigcap_{\lambda\in\Delta}\ker\lambda\backslash\bigcup_{\beta\in\Sigma^+\backslash\Delta}\ker\beta$.) Then $S(\Delta) \subset S(\Delta')$ if and only if $\Delta \subset \Delta'$.

COROLLARY 2.3: *The conjugacy classes of maximal (proper) parabolics of G are in one-to-one correspondence with elements of Π, the correspondence being*

$$\lambda \longleftrightarrow S(\Pi\backslash\{\lambda\}).$$

Example 2.4: $G = \mathrm{SU}(2,2)$. We describe this example in some detail as it sheds light on the results to follow. We consider G as the group of unimodular automorphisms of the hermitian form

$$z_1\overline{z}_4 + z_2\overline{z}_3 + z_3\overline{z}_2 + z_4\overline{z}_1.$$

With respect to the basis z_1, z_2, z_3, z_4, the matrix of this Hermitian form is

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The group G is the set $\{g \in \mathrm{SL}(4,\mathbb{C}) | g^*Qg = Q\}$, and its Lie algebra is

$$\mathfrak{g} = \{X \in \mathfrak{sl}(4,\mathbb{C}) | X^*Q + QX = 0\}.$$

Let $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. If we write $X \in \mathfrak{g}$ in the form $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A, B, C, D \in \mathfrak{gl}(2,\mathbb{C})$, then

$$\mathfrak{g} = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} | D = -sA^*s,\ B = -sB^*s,\ C = -sC^*s,\ \mathrm{Re}(\mathrm{tr}(A)) = 0\}$$

The transformation $A \rightarrow sA^ts$ is reflection in the antidiagonal. The automorphism $\theta(X) = -X^*$ is a Cartan involution of $\mathfrak{g}$ with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \{\begin{pmatrix} A & B \\ -B^* & sAs \end{pmatrix} | A^* = -A,\ B = -sB^*s,\ \mathrm{tr}(A) = 0\}$$

$$\mathfrak{p} = \{\begin{pmatrix} A & B \\ B^* & -sAs \end{pmatrix} | A^* = A,\ B = -sB^*s\}$$

The maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is

$$\mathfrak{a} = \{\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & -x \end{pmatrix} | x, y \in \mathbb{R}\}$$

Define linear functionals α and β on $\mathfrak{a}$ by

$$\alpha(\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & -x \end{pmatrix}) = x - y \qquad \beta(\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & -x \end{pmatrix}) = 2y.$$

Then $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$. The set $\Pi = \{\alpha, \beta\}$ is a simple system of roots. The root spaces $\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_{\pm(\alpha+\beta)}$ are two dimensional, and the spaces $\mathfrak{g}_{\pm\beta}, \mathfrak{g}_{\pm(2\alpha+\beta)}$ are one dimensional. The positive root spaces are

$$\mathfrak{g}_\alpha = \{\begin{pmatrix} 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\overline{w} \\ 0 & 0 & 0 & 0 \end{pmatrix} | w \in \mathbb{C}\} \qquad \mathfrak{g}_{\alpha+\beta} = \{\begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 0 & 0 & -u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | u \in \mathbb{C}\}$$

$$\mathfrak{g}_\beta = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | x \in \mathbb{R}\} \qquad \mathfrak{g}_{2\alpha+\beta} = \{\begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | x \in \mathbb{R}\}$$

According to lemma 2.2, there are four conjugacy classes of parabolic subgroups of G corresponding to the four sets $\Pi, \{\alpha\}, \{\beta\}, \emptyset$. The parabolic $S(\emptyset)$ is a minimal parabolic and consists of upper triangular matrices in $\mathfrak{g}$. The parabolic corresponding to Π is G. The other two parabolics are proper and maximal. We will describe them. Let

$$H_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } H_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} \cdot H_\alpha$ and $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] = \mathbb{R} \cdot H_\beta$.

$S(\alpha) = S(H_\alpha + 2H_\beta)$: The components of the Langlands decomposition are

$$\begin{aligned} \mathfrak{m}(\alpha) &= \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathbb{C} \cdot H_\alpha \cong \mathfrak{sl}(2, \mathbb{C}) \\ \mathfrak{a}(\alpha) &= \mathbb{R} \cdot (H_\alpha + 2H_\beta) \\ \mathfrak{n}(\alpha) &= \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta}. \end{aligned}$$

Thus $\dim S(\alpha) = \dim \mathfrak{m}(\alpha) + \dim \mathfrak{a}(\alpha) + \dim \mathfrak{n}(\alpha) = 6 + 1 + 4 = 11$. Note that $S(\alpha)$ is the stabilizer of the flag $\{z_1, z_2\} \subset \mathbb{C}^4$.

$S(\beta) = S(H_\alpha + H_\beta)$: The components of the Langlands decomposition are

$$\begin{aligned} \mathfrak{m}(\beta) &= \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta} \oplus \mathbb{R} \cdot H_\beta \oplus i\mathbb{R} \cdot H_\alpha \\ \mathfrak{a}(\beta) &= \mathbb{R} \cdot (H_\alpha + H_\beta) \\ \mathfrak{n}(\beta) &= \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{2\alpha+\beta}. \end{aligned}$$

Thus $\dim S(\beta) = 4 + 1 + 5 = 10$. This group is the stabilizer of the flag $\{z_1\} \subset \{z_1, z_2, z_3\} \subset \mathbb{C}^4$. It is also the stabilizer in G of the **minimal flag** $\{z_1\} \subset \mathbb{C}^4$, since the orthogonal complement with respect to Q of z_1 is the span of $\{z_1, z_2, z_3\}$ and any element which fixes the line through z_1 also fixes its orthogonal complement.

It follows that the minimum codimension of a parabolic subgroup of G is $\dim G - \dim S(\alpha) = 4$. Note that $\mathbb{R}$-rank$(G) = 2$, and that the split real form of G is $\mathrm{SL}(4, \mathbb{R})$, which has real rank 3 and acts on a compact manifold of dimension 3. The maximal compact subgroup of G is $K = S(U(2) \times U(2))$, which has dimension 7, and which may act effectively only in dimension ≥ 4.

Let G be a connected noncompact semisimple real Lie group with finite center. There is a natural topology on G associated to the Zariski topology on the real algebraic group $\mathrm{Ad}(G)$. We call this the Zariski topology on G. In order to apply Corollary 1.13 we need to know what cocompact Zariski closed subgroups of semisimple groups look like. We recall first of all a result of Witte. For a parabolic $S = MAN$ we write $M^0 = L \cdot E$, where L is the product of all the noncompact simple factors of M^0 and E is the maximal compact normal subgroup of M^0. The decomposition $S^0 = LEAN$ is called the **refined Langlands decomposition** of S^0.

THEOREM 2.5 [19]: *Let G be a connected, semisimple Lie group with finite center. Let H be a closed cocompact subgroup of G. Then there is a parabolic S in G with refined Langlands decomposition $S^0 = LEAN$, a connected normal subgroup X of L, and a connected, closed subgroup Y of E such that (a) H is contained in S, and (b) $H^0 = XYN$.*

An immediate consequence is

COROLLARY 2.6: *Let G be a connected, semisimple Lie group with finite center, and H a cocompact Zariski closed subgroup of G. Then there is a parabolic subgroup $S^0 = LEAN$ of G and a closed, connected subgroup Y of E such that $H^0 = LYAN$.*

Proof: Choose S as in Witte's theorem. Then H is a cocompact subgroup of S. But H has finitely many components, so $H^0 = XYN$ is a cocompact subgroup of S^0. Clearly, we must have $X = L$. ∎

Definition 2.7: Let G be a connected semisimple Lie group with finite center. Define $h(G)$ to be the minimum codimension of a cocompact subgroup of G which does not contain any nontrivial proper connected normal subgroup of G.

COROLLARY 2.8: *Let G be a connected noncompact simple Lie group with finite center. Then $h(G)$ is the minimum codimension of a proper parabolic subgroup of G.*

COROLLARY 2.9: *Let G be a noncompact simple Lie group with finite center, and suppose G acts almost effectively on a compact manifold M. Let $x \in M$ be a point contained in a minimal set of the action. Let G_x be the stabilizer of x, and G_x^0 its identity component. Then there is a parabolic subgroup S of G with refined Langlands decomposition $S^0 = LEAN$ and a closed connected subgroup Y of E such that $N_G(G_x^0) = LYAN$. In particular, $N_G(G_x^0)$ is a cocompact subgroup of a parabolic in G.*

Proof: Let x be a point in a minimal set of the action. Then by corollary 1.12, $N_G(G_x^0)$ is a cocompact algebraic subgroup of G, and therefore has the desired form by corollary 2.6. ∎

Note that a parabolic subgroup of a semisimple Lie group contains all compact simple factors of the group. For this reason, corollary 2.8 is false for semisimple groups with compact factors. The following proposition describes the situation for a general semisimple group.

PROPOSITION 2.10: *Let G be a connected semisimple Lie group with finite center. Write $G = (\prod_{i=1}^{k} G_i)\cdot H$, where $G_1, \ldots, G_k$ are connected simple noncompact normal subgroups of G and H is the maximal normal compact subgroup of G. Let Q be a connected, cocompact subgroup of G which does not contain any simple factor of G. Suppose that Q is maximal with respect to this property. Then there is a maximal parabolic subgroup $S \subset \prod_{i=1}^{k} G_i$ with refined Langlands decompositions $S^0 = LEAN$, and a closed subgroup Y of EH maximal with respect to the property that it contain no connected normal subgroup of H, such that $Q = LYAN$. If $H = \{e\}$ then $Q = \prod_{i=1}^{k} S_i^0$, where S_i is a maximal parabolic subgroup of G_i.*

Proof: By the corollary to Witte's theorem there is a parabolic S in G with refined Langland's decomposition $S^0 = LE'AN$ and a closed connected subgroup $Y \subset E'$ such that $Q = LYAN$. Any parabolic subgroup of G is a product of parabolic subgroups in the simple factors, so there are parabolic subgroups $S_i \subset G_i$, $i = 1, \ldots, k$, with refined Langland's decompositions $S_i^0 = L_iE_iA_iN_i$, such that $S = (\prod_{i=1}^k S_i) \cdot H$. Moreover,

$$L = (\prod_{i=1}^{k} L_i) \qquad E' = (\prod_{i=1}^{k} E_i) \cdot H$$
$$A = (\prod_{i=1}^{k} A_i) \qquad N = (\prod_{i=1}^{k} N_i).$$

Set $E = \prod E_i$. Suppose Y' is a closed subgroup of EH which contains Y and does not contain any simple factor of H. Then $Q' = LY'AN$ contains Q and does not contain any simple factor of G, so by maximality $Q = Q'$ and therefore $Y = Y'$. ∎

Example 2.11: Let $G = \mathrm{SO}(n, 1)$ and let S be the (unique up to conjugacy) proper parabolic subgroup of G. The refined Langlands decomposition of S^0 is given by $L = \{e\}$, $E \cong \mathrm{SO}(n-1)$, $A \cong \mathbb{R}_+^*$, and $N \cong \mathbb{R}^{n-1}$. Let $\pi : S^0 \to \mathrm{SO}(n-1)$ be the map which is trivial on AN and the identity on $E = \mathrm{SO}(n-1)$. Then $\mathrm{Id} \times \pi$ embeds P in $G \times \mathrm{SO}(n-1)$. The image is cocompact and maximal with respect to the property that it contain no normal subgroup. ∎

COROLLARY 2.12: *Let $G = \prod_{i=1}^k G_i$ be a connected semisimple Lie group with finite center and no compact factors. Then $h(G) = \sum_{i=1}^k h(G_i)$, and $h(G)$ is the minimum codimension of a parabolic subgroup of G which does not contain any simple factor of G.*

3. Complex parabolics

In this section we recall the definition of a **complex** parabolic subalgebra of a complex semisimple Lie algebra and spell out the relationship between real parabolics and complex parabolics. Most of this material is well-known and we have omitted most of the proofs.

Let G be a connected **complex** simple Lie group, $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ be the decomposition of $\mathfrak{g}$ into root spaces for $\mathfrak{h}$. Let B be the Killing form on $\mathfrak{g}$ and for each $\alpha \in \Sigma$ let H_α be the unique element of $\mathfrak{h}$ such that $B(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{g}$. Let $\mathfrak{h}_{\mathbb{R}}$ be the real subspace of $\mathfrak{h}$ spanned by $\{H_\alpha\}_{\alpha \in \Sigma}$. Then B is real and positive definite on $\mathfrak{h}_{\mathbb{R}}$, and in particular each $\alpha \in \Sigma$ is real and non-trivial on $\mathfrak{h}_{\mathbb{R}}$, and Σ is an abstract root system on $\mathfrak{h}_{\mathbb{R}}$. We fix a positive system of roots Σ^+.

A maximal solvable subalgebra of $\mathfrak{g}$ is called a **Borel subalgebra**, and any Borel subalgebra is conjugate by an element of G to the subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$. A complex subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is **parabolic** if it contains a Borel subalgebra of $\mathfrak{g}$. Again, by conjugating , we need only consider subalgebras which contain $\mathfrak{b}$. A connected subgroup of G is called **parabolic** if it is the normalizer of a parabolic subalgebra of $\mathfrak{g}$.

LEMMA 3.1: *Let $\mathfrak{g}$ be a complex simple Lie algebra. Then the set of complex parabolic subalgebras coincides with the set of real parabolic subalgebras of $\mathfrak{g}$ considered as a real Lie algebra.*

Let $\mathfrak{g}$ be a real Lie algebra. We denote by $\mathfrak{g}^{\mathbf{C}}$ the complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbf{C}$. Suppose now that $\mathfrak{g}$ is simple and noncompact. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$. Let $\mathfrak{b}$ be a maximal abelian subalgebra of $Z_{\mathfrak{k}}(\mathfrak{a})$. Then $\mathfrak{b} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$ with corresponding real form $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{b} \oplus \mathfrak{a}$. Let Σ be the root system of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h})$ and $\Sigma(\mathfrak{g}, \mathfrak{a})$ the restricted root system of $\mathfrak{g}$. Since $\mathfrak{a}$ is a subspace of $\mathfrak{h}_{\mathbb{R}}$, we may choose orderings on the dual spaces $\mathfrak{h}'_{\mathbb{R}}$ and $\mathfrak{a}'$ such that for $\lambda \in \mathfrak{h}'_{\mathbb{R}}$, if $\lambda|\mathfrak{a}$ is non-zero, then λ is positive if and only if $\lambda|\mathfrak{a}$ is positive. Fix such orderings and let Σ^+ and $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ be the positive roots in Σ and $\Sigma(\mathfrak{g}, \mathfrak{a})$, respectively. Let Π and $\Pi(\mathfrak{g}, \mathfrak{a})$ be the associated simple roots. For $\alpha \in \Sigma$, write $\overline{\alpha}$ for the restriction of α to $\mathfrak{a}$. Let

$$\begin{aligned} \Sigma|\mathfrak{a} &= \{\overline{\alpha} \mid \alpha \in \Sigma\}\backslash\{0\} \\ \Sigma^+|\mathfrak{a} &= \{\overline{\alpha} \mid \alpha \in \Sigma^+\}\backslash\{0\} \\ \Pi|\mathfrak{a} &= \{\overline{\alpha} \mid \alpha \in \Pi\}\backslash\{0\}. \end{aligned}$$

LEMMA 3.2 [13]: *With the conventions above,*

1. $\Sigma|\mathfrak{a} = \Sigma(\mathfrak{g}, \mathfrak{a})$
2. $\Sigma^+|\mathfrak{a} = \Sigma^+(\mathfrak{g}, \mathfrak{a})$
3. $\Pi|\mathfrak{a} = \Pi(\mathfrak{g}, \mathfrak{a})$

LEMMA 3.3: *Let $\mathfrak{g}$ be a real semisimple (noncompact) Lie algebra. Let $\mathfrak{s}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{s}^{\mathbf{C}}$ is a complex parabolic subalgebra of $\mathfrak{g}^{\mathbf{C}}$. Conversely, if $\mathfrak{s}$ is a real subalgebra of $\mathfrak{g}$ such that $\mathfrak{s}^{\mathbf{C}}$ is a parabolic subalgebra in $\mathfrak{g}^{\mathbf{C}}$, then $\mathfrak{s}$ is a parabolic in $\mathfrak{g}$.*

Proof: We may assume that $\mathfrak{s} = \mathfrak{s}(H_0)$, where $H_0 \in \mathfrak{a}$ and H_0 is non-negative with respect to $\Sigma^+|\mathfrak{a}$. We will show that $\mathfrak{s}(H_0)^{\mathbf{C}}$ contains the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Note first that for any $H \in \mathfrak{h}$, $\mathfrak{h} \subset Z_{\mathfrak{g}^{\mathbf{C}}}(H)$. Since $\mathfrak{s}(H_0) \supset Z_{\mathfrak{g}}(H)$ and $Z_{\mathfrak{g}^{\mathbf{C}}}(H) = (Z_{\mathfrak{g}}(H))^{\mathbf{C}}$ it follows that $\mathfrak{h} \subset \mathfrak{s}(H_0)$. Now let $\alpha \in \Sigma^+$. If $\alpha|\mathfrak{a} = 0$ then $\mathfrak{g}_\alpha^{\mathbf{C}} \subset Z_{\mathfrak{g}^{\mathbf{C}}}(\mathfrak{a}) \subset Z_{\mathfrak{g}^{\mathbf{C}}}(H_0)$. If $\alpha|\mathfrak{a} \neq 0$ then $\alpha|\mathfrak{a} \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ so $\mathfrak{g}_{\alpha|\mathfrak{a}} \subset \mathfrak{s}(H_0)$. But $\mathfrak{g}_\alpha^{\mathbf{C}} \subset (\mathfrak{g}_{\alpha|\mathfrak{a}})^{\mathbf{C}}$ and the claim follows.

To prove the converse we introduce some temporary notation. Let $\mathfrak{g}_0$ be a real semisimple noncompact Lie algebra and $\mathfrak{g}$ its complexification. Let τ be complex conjugation in $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$, i.e., if $X, Y \in \mathfrak{g}_0$, then $\tau(X + iY) = X - iY$. Let $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$, and $\mathfrak{b}$ a maximal abelian subalgebra of $Z_{\mathfrak{k}}(\mathfrak{a})$. Then $\mathfrak{b} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{h} = \mathfrak{a}^{\mathbf{C}} \oplus \mathfrak{b}^{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}$ with real form $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a} \oplus i\mathfrak{b}$. Let $\mathfrak{s}_0$ be a real subalgebra of $\mathfrak{g}_o$ and $\mathfrak{s}$ its complexification. Then $\tau(\mathfrak{s}) = \mathfrak{s}$. Suppose now that $\mathfrak{s}$ is a parabolic subalgebra of $\mathfrak{g}$.

It follows from lemma 3.1 that $\mathfrak{s}$ is a real parabolic subalgebra of $\mathfrak{g}$, i.e., there is some $H \in \mathfrak{h}_{\mathbb{R}}$ such that (up to inner automorphism) $\mathfrak{s} = \mathfrak{s}(H)$. We may write $H = H_1 + iH_2$ for $H_1 \in \mathfrak{a}$ and $H_2 \in \mathfrak{b}$. Let

$$\Delta = \{\lambda \in \Sigma(\mathfrak{g}, \mathfrak{h}_{\mathbb{R}}) | \, \lambda(H) > 0\}$$
$$\Delta' = \{\lambda \in \Sigma(\mathfrak{g}, \mathfrak{h}_{\mathbb{R}}) | \, \lambda(H_1) > 0\}^{\cdot}$$

Then $\mathfrak{s}(H) = Z_{\mathfrak{g}}(H) \oplus \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda$. We are going to show that $\Delta = \Delta'$. From this it follows that $\mathfrak{s}(H) = \mathfrak{s}(H_1)$, and therefore that $\mathfrak{s} = \mathfrak{s}(H_1) \cap \mathfrak{g}_0 = \mathfrak{s}_0(H_1)$, where $\mathfrak{s}_0(H_1)$ denotes the parabolic subalgebra of $\mathfrak{g}_0$ defined by H_1.

To prove that $\Delta = \Delta'$ we argue by contradiction. Suppose $\lambda \in \Delta \backslash \Delta'$. Then $\lambda(H) > 0$ but $\lambda(H_1) = 0$. It follows that

$$\tau(\lambda)(H) = \lambda(\tau(H)) = \lambda(H_1 - iH_2) = -\lambda(H) < 0.$$

Thus $\tau(\lambda) \notin \Sigma^+$. But $\tau(\mathfrak{s}(H)) = \mathfrak{s}(H)$, so $\tau(\Delta) = \Delta$. This yields a contradiction, so $\Delta \backslash \Delta' = \emptyset$. If $\lambda \in \Delta' \backslash \Delta$, then $\lambda(H_1) > 0$ and $\lambda(iH_2) = -\lambda(H_1)$. Then $\tau(\lambda)(H) = \lambda(H_1 - iH_2) > 0$, and again we have a contradiction. Thus $\Delta = \Delta'$.
∎

Definition 3.4: A simple real Lie algebra $\mathfrak{g}$ is said to be split over $\mathbb{R}$ if $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, i.e., if $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}$. ∎

LEMMA 3.5: *Let $\mathfrak{g}$ be a simple noncompact real Lie algebra, and $\mathfrak{g}^{\mathbb{C}}$ its complexification. The map $\mathfrak{s} \xrightarrow{\phi} \mathfrak{s}^{\mathbb{C}}$ gives an injective map from the set of conjugacy classes of parabolic subalgebras of $\mathfrak{g}$ to the set of conjugacy classes of parabolic subgroups of $\mathfrak{g}^{\mathbb{C}}$. This map is surjective if and only if $\mathfrak{g}$ is split over $\mathbb{R}$.*

Proof: According to the preceding lemma, the map ϕ is well-defined. For the proof of the theorem we maintain the notation of the preceding lemma.

Let $\mathfrak{b}$ be a maximal abelian subalgebra of $Z_{\mathfrak{k}}(\mathfrak{a})$. Then $\mathfrak{h} = (\mathfrak{b} \oplus \mathfrak{a})^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ with corresponding real form $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{b} \oplus \mathfrak{a}$. If $\mathfrak{s}(H)$ is a parabolic subalgebra of $\mathfrak{g}$, H non-negative, then $\mathfrak{s}(H)^{\mathbb{C}}$ is a parabolic of $\mathfrak{g}^{\mathbb{C}}$. Moreover, since $\mathfrak{s}(H) = Z_{\mathfrak{g}}(H) \oplus \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \mathfrak{g}_\lambda$, it follows that

$$\begin{aligned}
\mathfrak{s}(H)^{\mathbb{C}} &= (Z_{\mathfrak{g}}(H))^{\mathbb{C}} \oplus \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} (\mathfrak{g}_\lambda)^{\mathbb{C}} \\
&= Z_{\mathfrak{g}^{\mathbb{C}}}(H) \oplus \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} (\mathfrak{g}_\lambda)^{\mathbb{C}}.
\end{aligned}$$

In particular, if $\mathfrak{s}(H)^{\mathbb{C}} = \mathfrak{s}(H')^{\mathbb{C}}$, then $Z_{\mathfrak{g}^{\mathbb{C}}}(H) = Z_{\mathfrak{g}^{\mathbb{C}}}(H')$, so $\mathfrak{s}(H) = \mathfrak{s}(H')$, and ϕ is injective.

Now if $\mathfrak{g}$ has real rank n, then there are 2^n subsets of $\Pi(\mathfrak{g}, \mathfrak{a})$, and therefore 2^n distinct conjugacy classes of parabolic subalgebras of $\mathfrak{g}$. Let $m = \mathbb{R}$–rank$((\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}}) = \mathbb{C}$–rank$(\mathfrak{g}^{\mathbb{C}})$. Then $m \geq n$ and equality holds if and only if $\mathfrak{g}$ is split over $\mathbb{R}$. Since a complex parabolic is the same as a real parabolic of $(\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}}$, there are 2^m conjugacy classes of parabolics in $\mathfrak{g}^{\mathbb{C}}$. Thus ϕ is surjective if and only if $\mathfrak{g}$ is split over $\mathbb{R}$. ∎

Example 3.6: This is a continuation of example 2.12. Recall that $\mathfrak{g} = \mathfrak{su}(2,2)$, $\Pi(\mathfrak{g}, \mathfrak{a}) = \{\alpha, \beta\}$. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(4, \mathbb{C})$ be the $\mathbb{C}$-linear span of $\mathfrak{g}$ in $\mathfrak{gl}(4, \mathbb{C})$. The set of trace zero matrices in $\mathfrak{g}^{\mathbb{C}}$ is a Cartan subalgebra with real part

$$\mathfrak{h}_{\mathbb{R}} = \{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix} | x, y, z \in \mathbb{R}, x + y + z + w = 0 \}.$$

Define linear functionals γ, δ, and η on $\mathfrak{h}_{\mathbb{R}}$ by

$$\gamma(\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix}) = x - y \qquad \delta(\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix}) = y - z \qquad \eta(\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix}) = z - w.$$

Then $\overline{\gamma} = \overline{\eta} = \alpha$ and $\overline{\delta} = \beta$. The set $\Pi = \{\gamma, \delta, \eta\}$ is a simple system of roots for $\mathfrak{g}^{C}$. ∎

4. Maximal Parabolics

Our next goal is to determine the maximal parabolics of maximal dimension in the real simple Lie algebras. We do this first for complex simple Lie algebras. We then study the real non-complex simple Lie algebras, and use Satake diagrams to relate parabolics in the real algebra to parabolics in the complexification. This allows us to do all our computations in the complex simple Lie algebras, where they are significantly easier.

LEMMA 4.1: *Let $\mathfrak{g}$ be a real simple Lie algebra, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition, $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$, and Δ a subset of $\Pi(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{s}(\Delta)$ be the parabolic subalgebra associated to Δ and $\mathfrak{s}(\Delta) \oplus \mathfrak{a}(\Delta) \oplus \mathfrak{n}(\Delta)$ the Langlands decomposition. Then* $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}(\Delta)) = \dim \mathfrak{n}(\Delta) = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{m}(\Delta) - \dim \mathfrak{a}(\Delta))$.

Proof: Let θ be the Cartan involution of $\mathfrak{g}$ associated to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Recall that $\mathfrak{m}(\Delta) \oplus \mathfrak{a}(\Delta)$ is θ-stable, and $\theta(\lambda) = -\lambda$ for $\lambda \in \Sigma$. For $\lambda \in \Sigma \backslash \Sigma_H$, $\mathfrak{g}_\lambda \subset \mathfrak{n}_\Delta$ if and only if λ is positive. Thus

$$\mathfrak{g} = \theta(\mathfrak{n}_\Delta) \oplus \mathfrak{m}_\Delta \oplus \mathfrak{a}_\Delta \oplus \mathfrak{n}_\Delta$$

and the lemma follows immediately from this. ∎

PROPOSITION 4.2: *Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, Σ the root system of $(\mathfrak{g}, \mathfrak{h})$, Σ^+ a set of positive roots and $\Pi \subset \Sigma^+$ the simple roots. Let Δ be a subset of Π and $\mathfrak{s}_\Delta = \mathfrak{s}(\Delta)$ the associated parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{s}_\Delta = (\mathfrak{m}_\Delta \oplus \mathfrak{a}_\Delta) \ltimes \mathfrak{n}_\Delta$ be the Langlands decomposition. Then $\mathfrak{m}_\Delta$ is semisimple, and with respect to an appropriate ordering the set of simple roots of $(\mathfrak{m}_\Delta, \mathfrak{m}_\Delta \cap \mathfrak{h})$ is Δ. The Dynkin diagram of $\mathfrak{m}_\Delta$ is obtained from the diagram of $\mathfrak{g}$ by deleting all vertices not in Δ as well as all edges incident on such vertices.*

Proof: Choose $H \in \mathfrak{h}_{\mathbb{R}}$ such that $\Pi_H = \Delta$. Then $\mathfrak{m}_\Delta \oplus \mathfrak{a}_\Delta = Z_{\mathfrak{g}}(H) = \mathfrak{h} \oplus \sum_{\lambda \in \Sigma_H} \mathfrak{g}_\lambda$. The subalgebra $\mathfrak{m}_\Delta$ is the orthogonal complement in $Z_{\mathfrak{g}}(H)$ of $\mathfrak{a}_\Delta$. Thus $\mathfrak{m}_\Delta$ is θ-invariant, where θ is the Cartan involution of $\mathfrak{g}$ determined by $\mathfrak{h}$, so $\mathfrak{m}_\Delta$ is reductive. We may decompose $\mathfrak{m}_\Delta$ as

$$\mathfrak{m}_\Delta = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathfrak{z},$$

where $\mathfrak{z}$ is the center of $\mathfrak{m}_\Delta$. Since $\mathfrak{m}_\Delta$ commutes with $\mathfrak{a}_\Delta$ and $\mathfrak{h} \subset \mathfrak{m}_\Delta \oplus \mathfrak{a}_\Delta$, it follows that $\mathfrak{z}$ is $\mathfrak{h}$-invariant, and therefore decomposes as a direct sum of root spaces :

$$\mathfrak{z} = \sum_{\lambda \in \Gamma} \mathfrak{g}_\lambda.$$

But for $\lambda \in \Sigma_H$, $-\lambda$ is also in Σ_H, and $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \neq \{0\}$. It follows that $\mathfrak{z} = 0$ since the root spaces $\mathfrak{g}_\lambda$ are one-dimensional.

Order Σ_H by the induced ordering from Σ. It is easy to see that the simple roots in Σ_H^+ are the elements of Δ. The last assertion of the proposition now follows because Σ_H is a root system contained in Σ, and generated by Δ. ∎

Note that the proposition is not true for a real simple Lie algebra. Indeed, the restricted root spaces are not in general one-dimensional, and $\mathfrak{m}_\Delta$ is not in general semisimple. It is also much more difficult to compute the dimension of $\mathfrak{m}_\Delta$ in the real simple case. We will, however, use lemma 4.1 to determine the minimal codimension of a maximal parabolic in every real simple Lie group. First, we determine these codimensions for complex simple groups. For $\alpha \in \Pi$ we write $\mathfrak{s}_\alpha = \mathfrak{s}(\Pi \backslash \{\alpha\})$.

PROPOSITION 4.3: *The maximal parabolics of maximal dimension in the complex simple Lie algebras are listed in table 1, along with the complex codimension of these parabolics.*

Proof: We do a case by case analysis.

$\mathfrak{a}_\ell$: If we delete the root α_i from Π we are left with the root system $\mathfrak{a}_{i-1} \times \mathfrak{a}_{\ell-i}$, and the associated Lie algebra has dimension $(i^2-1)+(\ell-i+1)^2-1$. Thus $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_i}) = i(\ell-i+1)$. This expression is minimized when $i = 1$ or ℓ.

$\mathfrak{b}_\ell$: 1) Deleting α_i for $i < \ell - 1$ leaves $\mathfrak{a}_{i-1} \times \mathfrak{b}_{\ell-i}$, which has dimension $(i^2-1)+(\ell-i)(2(\ell-i)+1)$. Thus $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_i} = \frac{i}{2}(4\ell-3i+1)$. This is minimized for $i = 1$, in which case $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_i}) = 2\ell - 1$.

2) Deleting $\alpha_{\ell-1}$ leaves $\mathfrak{a}_{\ell-2} \times \mathfrak{a}_1$, and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_{\ell-1}}) = \frac{1}{2}(\ell^2 + 3\ell - 4)$ which is larger than $2\ell - 1$ for $\ell > 2$.

3) Deleting α_ℓ leaves $\mathfrak{a}_{\ell-1}$, and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_\ell}) = \frac{1}{2}(\ell^2 + \ell)$, which is larger than $2\ell - 1$ for $\ell > 2$, and equal to it for $\ell = 2$.

$\mathfrak{c}_\ell$: The computation is exactly the same as for $\mathfrak{b}_\ell$.

$\mathfrak{d}_\ell$: 1) Deleting α_i for $i < \ell - 2$ leaves $\mathfrak{a}_{i-1} \times \mathfrak{d}_{\ell-i}$, and

$$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_i}) = \frac{i}{2}(4\ell - 3i - 1).$$

This is minimized by $i = 1$, in which case $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_1}) = 2\ell - 2$.

2) Deleting $\alpha_{\ell-2}$ leaves $\mathfrak{a}_{\ell-1} \times \mathfrak{a}_1 \times \mathfrak{a}_1$ and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_{\ell-2}}) = \frac{1}{2}(\ell^2 - \ell - 6)$. This is larger than $2\ell - 2$ for $\ell > 4$ and equal for $\ell = 4$.

3) Deleting $\alpha_{\ell-1}$ or α_ℓ leaves $\mathfrak{a}_{\ell-1}$, and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_{\ell-1}}) = \operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_\ell}) = \frac{1}{2}(\ell^2 - \ell)$. This is larger than $2\ell - 2$ for $\ell \geq 4$.

$\mathfrak{e}_6$: 1) Deleting α_1 or α_6 leaves $\mathfrak{d}_5$, and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_i}) = 14$.

2) Deleting α_2 leaves $\mathfrak{a}_5$. $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_2}) = 21$.

3) Deleting α_3 or α_5 leaves $\mathfrak{a}_1 \times \mathfrak{a}_4$. $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_i}) = 25$.

4) Deleting α_4 leaves $\mathfrak{a}_2 \times \mathfrak{a}_1 \times \mathfrak{a}_2$, and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_4}) = 29$.

$\mathfrak{e}_7$:
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_1}) = 33$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_5}) = 50$
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_2}) = 42$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_6}) = 42$
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_3}) = 47$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_7}) = 27$
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_4}) = 53$

$\mathfrak{e}_8$:
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_1}) = 78$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_7}) = 83$
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_5}) = 104$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_4}) = 106$
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_2}) = 92$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_8}) = 57$
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_6}) = 97$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_3}) = 98$

$\mathfrak{f}_4$:
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_1}) = 15$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_2}) = 20$
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_3}) = 20$ $\quad$ $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_4}) = 15$

$\mathfrak{g}_4$:
$\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_1}) = \operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\alpha_2}) = 5$. ∎

We turn next to the determination of the maximal parabolics of maximal dimension in the real forms of the complex simple Lie algebras. The situation here is quite a bit more complicated than in the complex case because the Dynkin diagram of a restricted root system does not determine the associated real form. It is necessary also to list the multiplicities of the roots. Our approach to computing the maximal parabolics of a real form $\mathfrak{g}$ is to use the "Satake diagram" of $\mathfrak{g}$ to relate the maximal parabolics of $\mathfrak{g}$ to parabolics (not necessarily maximal) in $\mathfrak{g}^{\mathbf{C}}$, and then to do the necessary computations in $\mathfrak{g}^{\mathbf{C}}$.

Definition 4.4: Let $\mathfrak{g}$ be a real simple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. With the conventions we have employed above, the **Satake diagram** of $(\mathfrak{g}, \mathfrak{a})$ consists of

(1) The Dynkin diagram of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$.
(2) A coloring of the vertices of the diagram: black if the associated restricted simple root restricts to 0 on $\mathfrak{a}$, white otherwise.
(3) A "curved arrow" joining two white vertices if and only if the associated simple roots restrict to the same root on $\mathfrak{a}$. ∎

Example 4.5: Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{a} = \mathfrak{h}_{\mathbb{R}}$. Then the Satake diagram of $(\mathfrak{g}, \mathfrak{a})$ consists of two copies of the Dynkin diagram of $(\mathfrak{g}, \mathfrak{h})$, with all vertices colored white, and curved arrows connecting corresponding pairs of vertices in the two copies of the Dynkin diagram.

The Satake diagrams of the real forms of the classical complex groups were constructed by Satake [13], and for the exceptional groups by Araki [1]. A table of the Satake diagrams of the real non-complex simple Lie algebras and the multiplicities of the restricted roots is reproduced in Helgason [6, pp 532-534]. For ease of reference we have reproduced part of this table below (table 2). ∎

Definition 4.6: Let $\mathfrak{g}$ be a simple real Lie algebra. A simple root $\alpha \in \Pi(\mathfrak{g}, \mathfrak{a}) = \Pi \mid \mathfrak{a}$ is said to **split** if there are two simple roots $\alpha_1, \alpha_2 \in \Pi(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$ such that $\overline{\alpha}_1 = \overline{\alpha}_2 = \alpha$. ∎

LEMMA 4.7: *Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra, $\alpha \in \Pi(\mathfrak{g}, \mathfrak{a})$, and $\mathfrak{s}_\alpha = \mathfrak{s}(\Pi(\mathfrak{g}, \mathfrak{a}) \backslash \{\alpha\})$ the corresponding maximal parabolic of $\mathfrak{g}$.*

(1) *If α does not split, then $(\mathfrak{s}_\alpha)^{\mathbf{C}}$ is a maximal parabolic of $\mathfrak{g}^{\mathbf{C}}$.*
(2) *If α splits, then $(\mathfrak{s}_\alpha)^{\mathbf{C}} = \mathfrak{s}_{\alpha_1, \alpha_2} = \mathfrak{s}(\Pi \backslash \{\alpha_1, \alpha_2\})$, where $\overline{\alpha}_1 = \overline{\alpha}_2 = \alpha$.*
(3) $\operatorname{codim}_{\mathbb{R}}(\mathfrak{g} : \mathfrak{s}_\alpha) = \operatorname{codim}_{\mathbb{C}}(\mathfrak{g}^{\mathbf{C}} : (\mathfrak{s}_\alpha)^{\mathbf{C}})$.

Proof: The proof is a straightforward application of results we established in §3 and is left to the reader. ∎

THEOREM 4.8: *The parabolics of maximum dimension in the real non-complex simple groups are listed in table 2, along with the Satake diagrams and the codimension of the maximal parabolics of maximum dimension.*

Proof: Let $\alpha \in \Pi$ be a root such that $\mathfrak{s}_\alpha \subset \mathfrak{g}^{\mathbf{C}}$ is a complex parabolic of maximal dimension. If $\overline{\alpha}$ does not split then $(\mathfrak{s}_{\overline{\alpha}})^{\mathbf{C}} = \mathfrak{s}_\alpha$, and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\overline{\alpha}}) =$

$\operatorname{codim}(\mathfrak{g}^{\mathbf{C}} : \mathfrak{s}_\alpha)$. By inspecting the list of Satake diagrams one sees that the theorem follows immediately from lemma 4.1 and proposition 4.3 for the cases $\mathfrak{sl}(n+1,\mathbf{R})$, $\mathfrak{so}(p,q)$, $\mathfrak{sp}(n,\mathbf{R})$, $\mathfrak{e}_{6(6)}$, $\mathfrak{e}_{6(-26)}$, $\mathfrak{e}_{7(7)}$, $\mathfrak{e}_{7(-25)}$, $\mathfrak{e}_{8(8)}$, $\mathfrak{e}_{8(-24)}$, $\mathfrak{f}_{4(4)}$, $\mathfrak{f}_{4(-20)}$, and $\mathfrak{g}_{2(2)}$. Similarly, if we consider the cases in which no restricted root splits we need only compute the codimensions of maximal complex parabolics in $\mathfrak{g}^{\mathbf{C}}$ corresponding to roots in the Satake diagram which are colored white. These are the Lie algebras $\mathfrak{sl}(n+1,\mathbf{H})$, $\mathfrak{sp}(p,q)$, $\mathfrak{so}^*(2n)$ (n even), and $\mathfrak{e}_{7(-5)}$. Referring to the computations in the proof of proposition 4.3 one verifies the theorem in these four cases.

There are six cases remaining; we treat them one by one.

$\mathfrak{g} = \mathfrak{su}(p,1)$. There is only one parabolic, $\mathfrak{s}_{\lambda_1}$, and $(\mathfrak{s}_{\lambda_1})^{\mathbf{C}} = \mathfrak{s}_{\alpha_1,\alpha_p}$. The codimension of $\mathfrak{s}_{\alpha_1,\alpha_p}$ in $\mathfrak{g}^{\mathbf{C}}$ is $\frac{1}{2}(\dim(\mathfrak{g}) - \dim(\mathfrak{a}_{p-2}) - 2) = 2p-1$.

$\mathfrak{g} = \mathfrak{so}^*(2n), n \text{ odd}, n > 3$. The only root of $(\mathfrak{g},\mathfrak{a})$ which is split is λ_{n-1}, and $(\mathfrak{s}_{\lambda_{n-1}})^{\mathbf{C}} = \mathfrak{s}_{\alpha_{n-1},\alpha_n}$, which is contained in $\mathfrak{s}_{\alpha_n}$. Now $\dim(\mathfrak{s}_{\alpha_n}) < \dim(\mathfrak{s}_{\alpha_2})$, so $\mathfrak{s}_{\lambda_2}$ is the maximal parabolic of maximal dimension in $\mathfrak{g}$.

$\mathfrak{g} = \mathfrak{e}_{6(2)}$. $(\mathfrak{s}_{\lambda_1})^{\mathbf{C}} = \mathfrak{s}_{\alpha_1,\alpha_6}$ and $\mathfrak{s}_{\alpha_1,\alpha_6}$ has codimension $\frac{1}{2}(\dim(\mathfrak{g}) - \dim(\mathfrak{d}_4) - 2) = \frac{1}{2}(78-28-2) = 24$. $(\mathfrak{s}_{\lambda_3})^{\mathbf{C}} = \mathfrak{s}_{\alpha_3,\alpha_5} \subset \mathfrak{s}_{\alpha_3}$, which has codimension 25. But $(\mathfrak{s}_{\lambda_2})^{\mathbf{C}} = \mathfrak{s}_{\alpha_2}$ has codimension 21, so $\mathfrak{s}_{\lambda_2}$ is of maximal dimension.

$\mathfrak{g} = \mathfrak{e}_{6(14)}$. As in the preceding case, $(\mathfrak{s}_{\lambda_1})^{\mathbf{C}}$ has codimension 24 and $(\mathfrak{s}_{\lambda_2})^{\mathbf{C}} = \mathfrak{s}_{\alpha_2}$ has codimension 21.

$\mathfrak{g} = \mathfrak{su}(p,q)$ $p > q > 1$. $(\mathfrak{s}_{\lambda_i})^{\mathbf{C}} = \mathfrak{s}_{\alpha_i,\alpha_{p+q-i}}$. Deleting α_i and α_{p+q-i} from the root system of $\mathfrak{a}_{p+q-1}$ leaves $\mathfrak{a}_{i-1} \times \mathfrak{a}_{i-1} \times \mathfrak{a}_{p+q-2i-1}$. Thus $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\lambda_i}) = 2i(p+q) - 3i^2$. This is minimized for $i=1$, and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\lambda_1}) = 2(p+q) - 3$.

$\mathfrak{g} = \mathfrak{su}(q,q)$. For $i < q$ the same computation applies as in the preceding case. The root λ_q is not split, and $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\lambda_q}) = \operatorname{codim}(\mathfrak{g}^{\mathbf{C}} : \mathfrak{s}_{\alpha_q}) = q^2$. But $\operatorname{codim}(\mathfrak{g} : \mathfrak{s}_{\lambda_1}) = 4q-3$, which is less than $q(q+1)$ for $q > 2$, so $\mathfrak{s}_{\lambda_1}$ is of maximal dimension for $q > 2$. For $q = 2$, $\mathfrak{s}_{\lambda_2}$ is maximal. ∎

5. Low dimensional representations

For the proofs of the theorems in §6 it will be necessary to know something about the linear isotropy representation of G at a fixed point of the action. Specifically, we need to know the minimum dimension of a faithful real representation of a real simple Lie algebra. We consider this question in this section.

Let $\mathfrak{g}$ be a complex simple Lie algebra and λ an **algebraically integral** linear functional on $\mathfrak{h}$, i.e., $\lambda \in \mathfrak{h}'$, and $\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha \in \Sigma$. Let $\Pi = \{\alpha_1,\dots,\alpha_k\}$ be the set of simple roots, and define $\lambda_i = \frac{2\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}$. The linear functional λ is said to be **dominant** if $\lambda_i \geq 0$ for $i = 1,\dots,k$. The following classical theorem asserts that the complex irreducible representations of $\mathfrak{g}$ are in one-to-one correspondence with ordered k-tuples of non-negative integers $\lambda = (\lambda_1,\dots,\lambda_k)$.

THEOREM (Theorem of the highest weight): *Let $\mathfrak{g}$ be a complex simple Lie algebra. The irreducible complex representations of $\mathfrak{g}$ are in one-to-one correspondence with the dominant, algebraically integral linear functionals on $\mathfrak{h}$, the correspondence being that λ is the highest weight of the representation ρ_λ with respect to the ordering on $\mathfrak{h}'$.*

LEMMA 5.1: *Let λ and λ' be dominant, algebraically integral linear functionals on $\mathfrak{h}$, and suppose $\lambda_i \geq \lambda'_i$ for $i = 1,\dots,k$. Then for all $\alpha \in \Sigma^+$, $\langle\lambda,\alpha\rangle \geq \langle\lambda',\alpha\rangle$.*

Proof: The difference $\lambda - \lambda'$ is a dominant, algebraically integral linear functional on $\mathfrak{h}$. The lemma then follows from the fact that for a dominant, algebraically integral functional λ, $\langle\lambda,\alpha\rangle \geq 0$ for all $\alpha \in \Sigma^+$. (cf. [8, 4.15]). ∎

THEOREM (Weyl dimension formula): *The dimension of ρ_λ is*

$$d_\lambda = \prod_{\alpha\in\Sigma^+} \frac{\langle\lambda+\delta,\alpha\rangle}{\langle\delta,\alpha\rangle},$$

where $\delta = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$.

COROLLARY 5.2: *Let λ and λ' be dominant, algebraically integral linear functionals on $\mathfrak{h}$. If $\lambda_i \geq \lambda'_i$ for $i = 1,\dots,k$, then $d_\lambda \geq d_{\lambda'}$, and equality holds if and only if $\lambda = \lambda'$.*

Proof: By lemma 5.1, $\langle\lambda,\alpha\rangle \geq \langle\lambda',\alpha\rangle$ for all $\alpha \in \Sigma^+$, and since $\langle\delta,\alpha\rangle > 0$ for all $\alpha \in \Sigma^+$, it follows that $\langle\lambda+\delta,\alpha\rangle \geq \langle\lambda'+\delta,\alpha\rangle > 0$ for all $\alpha \in \Sigma^+$, and if $\lambda \neq \lambda'$, then strict inequality holds for some α. Now it follows immediately from the Weyl dimension formula that $d_\lambda \geq d_{\lambda'}$, and equality holds if and only if $\lambda = \lambda'$. ∎

Definition 5.3: The i-th **fundamental representation** ρ_i of $\mathfrak{g}$ is the complex representation with highest weight η_i, where $\frac{2\langle\eta_i,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} = 1$ and $\langle\eta_i,\alpha_j\rangle = 0$ for $i \neq j$. ∎

COROLLARY 5.4: *Let ρ_λ be an irreducible complex representation of the complex simple Lie algebra $\mathfrak{g}$ with highest weight λ. If ρ_λ is not fundamental, then*

$$d_\lambda \geq \min(\{\max(d_{\eta_i}, d_{\eta_j})\}_{i \neq j} \cup \{d_{2\eta_i}\}_{i=1,\ldots,k}).$$

Proof: If ρ_λ is not fundamental then there are either two indices i and j such that $\lambda_i > 0$ and $\lambda_j > 0$, or an index i such that $\lambda_i \geq 2$. By the preceding corollary, in the first case $d_\lambda \geq d_{\eta_i}$ and $d_\lambda \geq d_{\eta_j}$, and in the second case, $d_\lambda \geq d_{2\eta_j}$. ∎

LEMMA 5.5: *For the complex simple Lie algebras the minimum dimension of a real representation is given in table 1.*

Proof: The (complex) dimensions of the fundamental representations of the simple complex Lie algebras are listed, for example, in table 30 of [5]. None of the fundamental representations is real. By inspection of the table in [5] and corollary 5.4 one verifies the lemma. ∎

LEMMA 5.6: *For $n > 2$, $\mathfrak{sl}(n, \mathbb{H})$ has no real representation in dimension less than $4n$. $\mathfrak{sl}(2, \mathbb{H})$ has no real representation in dimension less than six.*

Proof: $\mathfrak{sl}(n, \mathbb{H})$ is a real form of $\mathfrak{sl}(2n, \mathbb{C})$, and the maximal compact subalgebra of $\mathfrak{sl}(n, \mathbb{H})$ is $\mathfrak{sp}(n)$. The fundamental representations of $\mathfrak{sl}(2n, \mathbb{C})$ have dimensions $d_{\eta_i} = \binom{2n}{i}, i = 1, \ldots, 2n-1$. It follows from the preceding corollary that for any irreducible representation ρ_λ of $\mathfrak{sl}(2n, \mathbb{C})$, either

(1) $\lambda = \eta_1$ or $\lambda = \eta_{2n-1}$,
(2) $\lambda = \eta_1 + \eta_{2n-1}$,
(3) $\lambda_1 \geq 2$ or $\lambda_{2n-1} \geq 2$, or
(4) $d_\lambda \geq \binom{2n}{2}$.

In case (1), ρ_{η_1} is equivalent to $\rho_{\eta_{2n-1}}$, and ρ_λ is the standard representation of $\mathfrak{sl}(2n, \mathbb{C})$ on $\mathbb{C}^{2n}$. In case (2), the representation ρ_λ is the adjoint representation which has dimension $4n^2-1$. In case (3), we have $d_\lambda \geq d_{2\eta_1}$, and a straightforward computation shows that $d_{2\eta_1} = (n+1)(2n+1)$, which is greater than or equal to $4n$ for $n > 2$. In case (4) we have $d_\lambda \geq 6$ for $n = 2$, and $d_\lambda \geq n(2n-1) > 4n$ for $n > 2$.

Now let π be an irreducible real representation of $\mathfrak{sl}(n, \mathbb{H})$ on a real vector space V. Then $\pi^{\mathbb{C}}$ is a complex representation of $\mathfrak{sl}(2n, \mathbb{C})$ on $V^{\mathbb{C}}$, and $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{C}}(V^{\mathbb{C}})$. If $\pi^{\mathbb{C}}$ is reducible, then the representation (π, V) is equivalent to a complex representation of $\mathfrak{sl}(n, \mathbb{H})$ considered as a real representation. In this case $\dim_{\mathbb{R}}(V) \geq 4n$. If $\pi^{\mathbb{C}}$ is irreducible, then either $d_{\pi^{\mathbb{C}}} > 4n$ (for $n > 2$) or $\pi^{\mathbb{C}} = \rho_{\eta_1}$. The second possibility does not occur, since if $\pi^{\mathbb{C}} = \rho_{\eta_1}$, then $\pi^{\mathbb{C}} \mid \mathfrak{sp}(n)$ is the standard representation of $\mathfrak{sp}(n)$, and $\pi(\mathfrak{sp}(n)) \subset \mathfrak{sl}(2n, \mathbb{R})$ implies that $\mathfrak{sp}(n) \cong \mathfrak{sp}(n, \mathbb{R})$, which is nonsense. ∎

LEMMA 5.7: *For $p + q > 4$, $\mathfrak{su}(p,q)$ has no real representation in dimension less than $2(p+q)$. For $p+q = 4$, $\mathfrak{su}(p,q)$ has no real representation in dimension less than six. $\mathfrak{su}(2,1)$ has no real representation in dimension less than six.*

Proof: $\mathfrak{su}(p,q)$ is a real form of $\mathfrak{sl}(n, \mathbb{C})$ for $n = p+q$. We consider first the case $p + q > 4$. As in the preceding lemma we conclude that an irreducible complex representation of $\mathfrak{sl}(n, \mathbb{C})$ either has dimension greater than $2n$, or is equivalent to the standard representation ρ of $\mathfrak{sl}(n, \mathbb{C})$ on $\mathbb{C}^n$. The representation $\rho | \mathfrak{su}(p,q)$ leaves invariant a Hermitian form on $\mathbb{C}^n$ of signature (p,q), and if $\rho | \mathfrak{su}(p,q)$ were real, then it would leave invariant a real quadratic form of signature (p,q). Thus we would have a homomorphism $\mathfrak{su}(p,q) \rightarrow \mathfrak{so}(p,q)$, which is impossible because $\dim(\mathfrak{su}(p,q)) > \dim(\mathfrak{so}(p,q))$. It follows that for $p+q > 4$, the minimum dimension of a real representation is $2(p+q)$.

For $p + q = 4$, we find that a complex representation of $\mathfrak{sl}(n, \mathbb{C})$ either has dimension at least 6 or is standard. Since the standard representation of $\mathfrak{su}(p,q)$ is not real we conclude that the minimal dimension of a real representation of $\mathfrak{su}(3,1)$ or $\mathfrak{su}(2,2)$ is 6.

For the Lie algebra $\mathfrak{su}(2,1)$, there are exactly two (equivalent) fundamental representations of $\mathfrak{sl}(3, \mathbb{C})$. If ρ_λ is a non-trivial, non-fundamental irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ then either

(1) $\lambda_1 \geq 2$ or $\lambda_2 \geq 2$, or
(2) $\lambda_1 \geq 1$ and $\lambda_2 \geq 1$.

In the second case, d_λ is greater than the dimension of the adjoint representation, which is 8. To obtain a lower bound on d_λ in case (1), it suffices to compute the dimension of $\rho_{2\eta_1}$. An easy computation shows that $d_{2\eta_1} = 6$. The assertion of the lemma now follows just as in the case of $p + q \geq 4$. ∎

LEMMA 5.8: *The minimum dimension of a real representation of $\mathfrak{so}(p,q)$, $(p+q > 4)$, is $p+q$.*

Proof: $\mathfrak{so}(p,q)$ is a real form of $\mathfrak{so}(n, \mathbb{C})$, where $n = p + q$. The fundamental representations of $\mathfrak{so}(n, \mathbb{C})$ have dimensions $\binom{n}{i}$, for $i = 1, \ldots, [\frac{n-1}{2}] - 1$, and

$2^{[\frac{n-1}{2}]}$. For $n > 4$, we conclude that every irreducible complex representation of $\mathfrak{so}(n, \mathbb{C})$ has dimension at least n, and therefore every real representation of $\mathfrak{so}(p,q)$ has real dimension at least n. The representation corresponding to $i = 1$ is the standard representation, which is real and has dimension n. ∎

LEMMA 5.9: *For $n > 4$, $\mathfrak{so}^*(2n)$ has no real representation in dimension less than $4n$. $\mathfrak{so}^*(6)$ has no real representation in dimension less than 6 and $\mathfrak{so}^*(8)$ has no real representation in dimension less than 8.*

Proof: The last two assertions follow from preceding lemmas and the isomorphisms $\mathfrak{so}^*(6) \cong \mathfrak{su}(3,1)$ and $\mathfrak{so}^*(8) \cong \mathfrak{so}(6,2)$. Suppose then that $n > 4$. As in the preceding lemma, the fundamental representations of $\mathfrak{so}(2n, \mathbb{C})$ have dimensions $\binom{2n}{i}$, $i = 1, \dots, n-2$, and 2^{n-1}. For $n > 4$, $\binom{2n}{i} > 4n$ for $2 \le i \le n-2$, and $2^{n-1} \ge 4n$. One computes also that $d_{2\eta_1} = (2n-1)(n+1)$, which is greater than $4n$ for $n > 4$. Thus the only possible real representation of $\mathfrak{so}^*(2n)$ of dimension less than $4n$ is one which complexifies to the standard representation of $\mathfrak{so}(2n, \mathbb{C})$. Recall that

$$\mathfrak{so}^*(2n) = \{X \in \mathfrak{su}(n,n) \mid X^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X = 0\}.$$

Thus $\mathfrak{so}^*(2n)$ leaves invariant (in the standard representation) a Hermitian form of signature (n,n), so if $\mathfrak{so}^*(2n)$ is real then $\mathfrak{so}^*(2n) \subset \mathfrak{so}(n,n)$. These groups have the same dimension, and are not isomorphic, so $\mathfrak{so}^*(2n) \not\subset \mathfrak{so}(n,n)$, and the standard representation of $\mathfrak{so}^*(2n)$ is not real. The lemma is proved. ∎

LEMMA 5.10: *For $p+q > 2$, $\mathfrak{sp}(p,q)$ has no real representation in dimension less than $4(p+q)$.*

Proof: $\mathfrak{sp}(p,q)$ is a real form of $\mathfrak{sp}(n, \mathbb{C})$, the fundamental representations of which have dimensions $\binom{2n}{i} - \binom{2n}{i-2}$, for $i = 1, \dots, n$ (we define $\binom{2n}{-1} = 0$). For $n > 2$, $\binom{2n}{i} - \binom{2n}{i-2} > 4n$ for $i = 2, \dots, n$. The representation $\rho_{2\eta_1}$ has dimension $d_{2\eta_1} = n(2n+1)$ (in fact, $\rho_{2\eta_1}$ is the adjoint representation), and $d_{2\eta_1} > 4n$ for $n > 1$. Thus if ρ is a real representation of $\mathfrak{sp}(p,q)$ in dimension less than $4n$, then its complexification must be the standard representation of $\mathfrak{sp}(n, \mathbb{C})$. The standard representation of $\mathfrak{sp}(n, \mathbb{C})$ preserves a skew symmetric non-degenerate bilinear form on $\mathbb{C}^{2n}$, and if the restriction of this representation to $\mathfrak{sp}(p,q)$ is real, then $\mathfrak{sp}(p,q)$ preserves a skew-symmetric bilinear form on

$\mathbb{R}^{2n}$, i.e., $\mathfrak{sp}(p,q) \subset \mathfrak{sp}(n,\mathbb{R})$. Since $\mathfrak{sp}(p,q)$ and $\mathfrak{sp}(n,\mathbb{R})$ are not isomorphic and have the same dimension, there can be no such inclusion. Therefore $\mathfrak{sp}(p,q)$ for $p+q>2$ has no real representation in dimension less than $4(p+q)$. ∎

Note that the Lie algebra $\mathfrak{sp}(1,1)$ is isomorphic to $\mathfrak{so}(4,1)$.

Definition 5.11: Let G be a real Lie group. Define $\Phi(G)$ to be the minimum dimension of an almost faithful real representation of G. For a real Lie algebra $\mathfrak{g}$ let $\Phi(\mathfrak{g})$ be the minimum dimension of a faithful real representation of $\mathfrak{g}$. ∎

Note that $\Phi(G) \geq \Phi(\mathfrak{g})$.

LEMMA 5.12: *Let* $\mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{g}_i$, *where each* $\mathfrak{g}_i$ *is a simple real Lie algebra. Then* $\Phi(\mathfrak{g}) = \sum_{i=1}^{k} \Phi(\mathfrak{g}_i)$.

Proof: We argue by induction on k. For $k=1$ there is nothing to show. Let (ρ, V) be a faithful real representation of $\mathfrak{g}$, and let $\mathfrak{g}' = \bigoplus_{i=1}^{k-1} \mathfrak{g}_i$. Let W_0 be a subspace of V on which $\rho(\mathfrak{g}_k)$ acts nontrivially and irreducibly, and let W be the smallest $\rho(\mathfrak{g})$-invariant subspace of W containing W_0. Let $W^\perp$ be a $\rho(\mathfrak{g})$-invariant complement to W. Let $U = \mathcal{H}om_{\mathfrak{g}_k}(W, W_0)$. Then $\mathfrak{g}'$ acts on U since $\mathfrak{g}'$ commutes with $\mathfrak{g}_k$. Moreover, the representation of $\mathfrak{g}'$ on $U \oplus W^\perp$ is faithful. By the induction hypothesis, $\dim(U \oplus W^\perp) \geq \Phi(\mathfrak{g}') = \sum_{i=1}^{k-1} \Phi(\mathfrak{g}_i)$. On the other hand,

$$\begin{aligned} \dim(V) &= \dim(W^\perp) + \dim(W) \\ &= \dim(W^\perp) + \dim(W_0) \cdot \dim(U). \end{aligned}$$

If $\dim(U) = 1$ then $\mathfrak{g}'$ acts trivially on U, so $\dim(W^\perp) \geq \Phi(\mathfrak{g}')$ and $\dim(V) \geq \dim(W^\perp) + \dim(W_0) \geq \Phi(\mathfrak{g}') + \Phi(\mathfrak{g}_k)$. If $\dim(U) > 1$ then $\dim(W_0) \cdot \dim(U) > \dim(W_0) + \dim(U)$. ∎

6. Low dimensional actions

Definition 6.1: For any Lie group G let $n(G)$ be the minimum dimension of a closed manifold on which G acts almost effectively and smoothly. ∎

LEMMA 6.2: *For any connected semisimple Lie group* G, $n(G) < \Phi(G)$.

Proof: Let $\rho : G \to GL(n,\mathbb{R})$ be an almost faithful real representation of G. Then G acts via ρ on $\mathbb{R}P^{n-1}$. Let H be the kernel of the action of G on $\mathbb{R}P^{n-1}$.

Then H fixes every line in $\mathbb{R}^n$, so $\rho(H) \subset \{\pm I\}$. It follows that H is discrete, so the action of G on $\mathbb{R}P^{n-1}$ is almost effective.

COROLLARY 6.3: *Let G be a semisimple Lie group. Write $G = \prod_{i=1}^{k} G_i$, where each G_i is simple. Then $n(G) \leq \Phi(G) - k$.*

Proof: Let ρ_i be an almost faithful representation of G_i on $\mathbb{R}^{n_i}$. Then G acts almost effectively on $\prod_{i=1}^{k} \mathbb{R}P^{n_i-1}$, so $n(G) \leq \sum_{i=1}^{k}(n_i - 1)$. ∎

LEMMA 6.4: *Let G be a compact simple Lie group. Then $n(G)$ is the minimum codimension of a proper closed subgroup of G, i.e., $n(G) = h(G)$.*

Proof: This is obvious because any orbit is compact, and therefore an action in minimal dimension must be transitive. Any transitive (nontrivial) action is almost effective since G is simple. ∎

LEMMA 6.5: *Let $G = \prod_{i=1}^{k} G_i$ be a connected semisimple Lie group with finite center. Let Q be a closed connected subgroup of G of codimension not greater than $h(G) + 1$. Suppose $N_G(Q)$ is cocompact in G. If $N_G(Q)$ contains a simple factor of G then so does Q.*

Proof: We argue by induction on k. The case $k = 1$ is trivial. Suppose the lemma is true for $l < k$. Suppose $N_G(Q)$ contains G_1. Let $G' = \prod_{i=2}^{k} G_i$, and let $\pi : G \to G_1/(G' \cap G_1)$ be the natural map. Since G_1 normalizes Q, $G_1 \cap Q$ and $\pi(Q)$ are normal subgroups of G_1 and $G_1/(G_1 \cap G')$, respectively. If $Q \cap G_1 = G_1$ we are done. Suppose then that $Q \cap G_1 = \{e\}$. We consider the cases $\pi(Q) = \{e\}$ and $\pi(Q) = \pi(G_1)$ separately.

Suppose $\pi(Q) = \{e\}$. Then $Q \subset G'$. We will show that

(1) $\operatorname{codim}(G' : Q) \leq h(G') + 1$,
(2) $N_{G'}(Q)$ is cocompact in G', and
(3) $N_{G'}(Q)$ contains a simple factor of G'.

It will follow by induction that Q contains a simple factor of G'.

To verify (2) simply note that $G/N_G(Q) = G'/N'_G(Q)$. For (1) we have

$$1 + h(G') + h(G_1) \geq h(G) + 1 \geq \operatorname{codim}(G : Q) = \dim(G_1) + \operatorname{codim}(G' : Q),$$

from which it follows that

$$\operatorname{codim}(G' : Q) \leq 1 + h(G') + (h(G_1) - \dim(G_1)) < h(G'),$$

the last inequality following from the fact that for any simple Lie group H, $h(H) < \dim(H) - 1$. Now (3) follows from the inequality $\operatorname{codim}(G' : Q) < h(G')$ because $N_G(G_x^0)$ is cocompact in G and, by definition, $h(G')$ is the minimum codimension of a cocompact subgroup of G' which contains no simple factors of G'.

Suppose now that $\pi(Q) = \pi(G_1)$. Let $x \in G_1$ be an element not in the center and let x' be an element of G' such that $xx' \in Q$. Let h be an element of G_1 which does not commute with x. Then $hxx'h^{-1} = hxh^{-1}x' \in Q$ (since G_1 normalizes Q), so $hxh^{-1}x' \cdot (xx')^{-1} = hxh^{-1}x^{-1}$ is in Q. This contradicts the fact that $Q \cap G_1 = \{e\}$. ∎

THEOREM 6.6: *Let G be a noncompact simple Lie group with finite center. Then $n(G)$ is the minimum codimension of a maximal parabolic subgroup of G. If G acts almost effectively on a compact manifold M of dimension $n(G)$ then M is a finite equivariant covering of G/S for some maximal parabolic $S \subset G$.*

Proof: The first assertion is that $n(G) = h(G)$. Let S be a parabolic subgroup of G of codimension $h(G)$. Then G acts effectively on G/S, so $n(G) \leq h(G)$. Let G act almost effectively on a closed manifold M of dimension $n(G)$. Let x be a point in a minimal set of the action. Then $N_G(G_x^0)$ is a cocompact subgroup of a parabolic subgroup S. Since

$$h(G) \geq \dim(M) \geq \operatorname{codim}(G : G_x) \geq \operatorname{codim}(G : N_G(G_x^0)) \geq \operatorname{codim}(G : S),$$

it follows that either S is a proper parabolic of maximum dimension or $S = G$. In either case we must have $N_G(G_x^0) = S$.

Suppose $N_G(G_x^0) = G$. Then $G_x^0 = G$, i.e., x is a fixed point. Let ρ be the linear isotropy representation of G on $T_x(M)$. By lemma 6.2, for every simple Lie group G, $n(G) < \Phi(G)$. It follows that ρ is trivial. By a theorem of Stowe [15], every point in a neighborhood of x is a fixed point for G, and therefore G acts trivially on M.

We may assume then that $N_G(G_x^0) = S$, where S is a proper parabolic of maximum dimension in G. In this case we find that $\dim(M) = h(G)$ and $G_x^0 = S^0$. The parabolic S has only finitely many connected components so G_x is a subgroup of finite index in S. The G orbit through x is open since it is of full dimension in M, and compact since G/S is compact. It follows that G acts transitively on M, and

$$M \cong G/G_x \to G/N_G(G_x^0)$$

is a finite equivariant covering. ∎

Remark: It is often the case that for a simple non-compact Lie group G with finite center, $n(G) = n(K)$, where K is the maximal compact subgroup of G. However, this equality does not always hold. For example, for $G = \mathrm{SO}^*(2n)$, $n(G) = 4n - 7$ (for $n > 4$). The maximal compact subgroup of G is $U(n)$, and $n(U(n)) \leq 2n - 1$ since $U(n)$ acts effectively on S^{2n-1}. ∎

THEOREM 6.7: *Let G be a connected semisimple Lie group with finite center. Write $G = \prod_{i=1}^{k} G_i$, where $G_1, \ldots, G_k$ are the connected simple normal subgroups of G. Then $n(G) = h(G)$. If G acts almost effectively on a closed manifold M of dimension $n(G)$, then the action is transitive and the isotropy group is a cocompact subgroup of G which does not contain any simple factor of G. If G has no compact factors then the isotropy group is a subgroup of finite index in a parabolic subgroup of G, and $n(G) = \sum_{i=1}^{k} n(G_i)$.*

Proof: As in the preceding proof, we see that $n(G) \leq h(G)$. Let G act almost effectively on a closed manifold M of dimension $n(G)$. Let x be a point in a minimal set of the action. There are two possibilities:

(1) $N_G(G_x^0)$ is a cocompact subgroup of G containing a nontrivial connected normal subgroup of G, or
(2) $N_G(G_x^0)$ is a cocompact subgroup of G which contains no nontrivial connected normal subgroup of G.

If (2) holds, then arguing as in the proof of the previous theorem we conclude that G_x^0 is a subgroup of finite index in $N_G(G_x^0)$ and M is diffeomorphic to G/G_x^0. Thus to prove the theorem it suffices to show that (1) cannot hold.

We argue by induction on k. If $k = 1$ then G is simple and the theorem follows if G is compact from lemma 6.4 and if G is noncompact from the preceding theorem. Suppose the theorem is true for semisimple groups with fewer than k factors. We suppose that (1) holds and derive a contradiction. By lemma 6.5, G_x^0 contains a nontrivial connected normal subgroup of G, say G_1. Let $G' = \prod_{i=2}^{k} G_i$. Let N be the set of fixed points of G_1. Then N is a closed submanifold of M [15]. The submanifold N is G' invariant since G' commutes with G_1. Moreover, $\mathrm{codim}(M : N) \geq \Phi(G_1)$ since otherwise the isotropy representation of G_1 on the normal bundle of N is trivial, and by [15] the action of G_1 on M would be trivial. From the inequalities $\dim(M) \leq h(G)$ and $h(G_1) = n(G_1) < \Phi(G_1)$, it follows that

$$\dim(N) \leq \dim(M) - \Phi(G_1) < h(G) - h(G_1) \leq h(G').$$

By the induction hypothesis it follows that G' does not act almost effectively on N, i.e., there is a simple factor of G', say G_2, that pointwise fixes all of N. Let $G'' = \prod_{i=3}^{k} G_i$. Then repeating the argument we find that

$$\dim(N) < h(G) - h(G_1 G_2) \leq h(G''),$$

and therefore G'' does not act effectively on N. Continuing in this fashion we arrive at the conclusion that $\dim(N) < 0$, i.e., $N = \emptyset$. This is a contradiction, so (1) cannot hold. ∎

COROLLARY 6.8: *Let G be a semisimple Lie group with finite center. Then $\Phi(G) > h(G)$.*

Remark: If $G = \prod_{i=1}^{k} G_i$ has compact factors then it is not generally true that $n(G) = \sum_{i=1}^{k} n(G_i)$. For example, for $\mathfrak{g} = \mathfrak{so}(4)$ we have $n(\mathfrak{g}) = 3$. But $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, and $n(\mathfrak{so}(3)) = 2$. Although this example appears to be the only way the equality $n(G) = \sum n(G_i)$ can fail for semisimple groups, we will not pursue this question further. Note however that the inequality $n(G) \leq \sum n(G_i)$ always holds, even if the factors are not simple. ∎

THEOREM 6.9: *Let $G = \prod_{i=1}^{k} G_i$ be a connected semisimple Lie group with finite center acting on a closed manifold M of dimension $n(G) + 1$. Let K be a maximal compact subgroup of G. Then M/K is homeomorphic to S^1, I, or a point. In the first case the orbits of K fiber M over S^1.*

Proof: Let x be a point in a minimal set of the action. Then $N_G(G_x^0)$ is a cocompact subgroup of G. There are two possibilities:

(1) $N_G(G_x^0)$ is a cocompact subgroup of G containing no simple factor of G.

(2) $N_G(G_x^0)$ is a cocompact subgroup of G containing a simple factor of G.

Suppose (1) holds. Then

$$\begin{aligned}\operatorname{codim}(N_G(G_x^0) : G_x^0) &= \operatorname{codim}(G : G_x^0) - \operatorname{codim}(G : N_G(G_x^0)) \\ &\leq \dim(M) - h(G) = 1.\end{aligned}$$

If G_x^0 is not a cocompact subgroup of G then necessarily $\operatorname{codim}(N_G(G_x^0) : G_x^0) = 1$, and the orbit through x is of full dimension in M and noncompact, which contradicts Lemma 1.16. We may assume then that G_x^0 is a cocompact subgroup of G, of codimension $\leq h(G)+1$. We claim that K acts transitively on the G orbit through x. This suffices (by Mostert's theorem) to finish case (1), because the G orbit through x has codimension 0 or 1 in M.

Write $G = (\prod_{i=1}^{s} G_k) \cdot H$, where H is the maximal compact normal connected subgroup of G. Then using the notation of Proposition 2.10, $G_x^0 = LYAN$ for some closed subgroup $Y \subset EH$. Let K_i be the maximal compact subgroup of G_i. Then $K = (\prod_{i=1}^{s} K_i) \cdot H$, and since K_i acts transitively on $G_i/L_iA_iN_i$, it follows that K acts transitively in $G/LYAN$.

Now suppose (2) holds. Before proceeding with the proof, we describe an example which the reader should keep in mind. Take the direct sum of the standard representation of $\mathrm{SL}(2,\mathbb{R})$ with the trivial one-dimensional real representation, and consider the associated projective action on $\mathbb{R}P^2$. This action has exactly one fixed point. Now consider the action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}P^2 \times \mathbb{R}P^1$ obtained by taking the direct product of the previously described action with the standard action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}P^1$. The fixed point set of the normal subgroup $\{e\} \times \mathrm{SL}(2,\mathbb{R})$ is a copy of $\mathbb{R}P^1$.

To continue with the proof of the theorem, note that by lemma 6.5, G_x^0 contains a normal subgroup of G; let G_1 be the largest normal subgroup of G contained in G_x^0, and let $G' = Z_G(G_1)$ be the complementary subgroup. Let N be the set of G_1 fixed points. Then, reasoning as in the proof of 6.7, we conclude that the normal isotropy representation of G_1 is almost faithful, and therefore that

$$\dim N \leq \dim M - \Phi(G_1) \leq (h(G)+1) - (h(G_1)+1) \leq h(G').$$

If $\dim(N) < h(G')$, then we may argue as in the proof of 6.7 to obtain a contradiction. We may assume therefore that $h(G') = \dim N$, and thus that $h(G_1) + 1 = \Phi(G)$. By corollary 6.3, G_1 must be simple. Moreover, G' acts almost faithfully on N (since G_1 is the largest normal subgroup of G fixing x), and therefore by 6.6, G' acts transitively on N with cocompact stabilizer. It follows that K', the maximal compact subgroup of G', acts transitively on N.

Let K_1 be the maximal compact subgroup of G_1 (if G_1 is compact we are setting $K_1 = G_1$). Then K_1 pointwise fixes N and therefore acts trivially on $T_x(N)$. Let $V \subset T_x(M)$ be a K_1-invariant complement to $T_x(N)$. Note that $\dim V = h(G_1) = 1 = \Phi(G_1)$. We now require

LEMMA 6.10: *Let H be a simple Lie group with finite center and L a maximal compact subgroup. Let ρ be an almost faithful representation of H on a real vector space W of dimension $h(H) + 1$. Then the L-orbits in $W \backslash \{0\}$ are spheres of dimension $h(H)$.*

Proof: The action of H on the projective space $P(W)$ is transitive: this follows from 6.6 if H is noncompact and from the definition of $h(H)$ if H is compact. Moreover, in both cases L acts transitively on $P(W)$. Since L leaves invariant the metric spheres for some positive definite L-invariant bilinear form on W, it follows that the nontrivial L orbits are spheres of dimension $h(H)$. ∎

As remarked above, the normal isotropy representation of G_1 at x is almost faithful. From 6.10 it follows that the nontrivial K_1 orbits in V have dimension $h(G_1)$. We now apply the differentiable slice theorem for actions of compact groups [11]. It follows from this theorem that in a neighborhood of N the dimension of the orbits of $K = K_1 \cdot K'$ is the dimension of N (the K orbit of x) plus the dimension of the non-trivial orbits in the normal isotropy representation of K_x. The former is $h(G')$ and the latter is $h(G_1)$ (because $K_1 \subset K_x$). Thus there are K orbits of dimension $h(G') + h(G_1) = h(G) = \dim M - 1$. Now we can apply Mostert's theorem to conclude the proof. ∎

Remark: The proof of the theorem shows that if G acts almost effectively on a compact manifold of dimension less than $\Phi(G)$, then there are no fixed points for the action. Note that $\Phi(G) = n(G) + 1$ for $G = \mathrm{SL}(n, \mathbb{R})$ and $G = Sp(n, \mathbb{R})$. In all other cases $\Phi(G) > n(G) + 1$. Both $\mathrm{SL}(n, \mathbb{R})$ and $Sp(n, \mathbb{R})$ have actions in dimension $n(G) + 1$ with fixed points. ∎

THEOREM 6.11: *Let G be a connected semisimple Lie group with finite center acting almost effectively on a closed manifold M of dimension $n(G) + 1$. Let M/G be the orbit space of the action with the quotient topology. Then M/G is obtained from S^1 or I by identifying possibly infinitely many connected, open subsets to points. In particular, every G orbit is either compact or open.*

Proof: If G acts transitively the assertion is immediate. Otherwise, M/K is homeomorphic to S^1 or I. There is a natural quotient map $M/K \xrightarrow{q} M/G$ which commutes with the quotient maps $p : M \to M/K$ and $s : M \to M/G$. The G-orbits in M are connected, K-saturated subsets of M, so for any $x \in M$, $p(G \cdot x)$ is a connected subset of M/K. If $p(G \cdot x)$ is not a single point then it contains an open subset of M/K and therefore $G \cdot x$ contains an open subset of M, so $G \cdot x$ is open. Thus $p(G \cdot x)$ is an open connected subset of M/K, and $q(p(G \cdot x)) = s(G \cdot x)$ is a single point in M/G. ∎

In the next section we construct, for any noncompact semisimple Lie group G, a smooth action of G on a closed manifold of dimension $n(G)+1$ with infinitely many open orbits. The following lemma shows that the open orbits may not accumulate on a fixed point of the action.

LEMMA 6.12: *Let G be a noncompact semisimple Lie group acting smoothly and almost effectively on a closed manifold M of dimension $n(G)+1$. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence of points in M contained in distinct G orbits, and suppose the orbit of each x_i is open. Then every limit point of $\{x_i\}_{i \in \mathbb{N}}$ is contained in a compact G orbit of dimension $n(G)$.*

Proof: By Theorem 6.11, one sees that if x is a limit point of $\{x_i\}$, then either x is contained in a compact codimension one G–orbit (which coincides with a K–orbit) or x is a G–fixed point. We will show that the latter is not possible.

Recall from the proof of 6.9 that if x is a fixed point then G is locally isomorphic to either $\mathrm{SL}(n, \mathbb{R})$ or $Sp(n, \mathbb{R})$, and the isotropy representation ρ of G on $T_x(M)$ is the "standard" representation. Let $\mathfrak{h}$ be a split Cartan subalgebra of $\mathfrak{g}$, and H the connected subgroup of G with Lie algebra $\mathfrak{h}$. Then $\rho(H)$ has no invariant lines, i.e., 1 is not an eigenvalue of $\rho(H)$. Then by [7], x is a stable, isolated fixed point of H. On the other hand, there is a sequence of closed G orbits converging to x (for each i choose a closed G orbit separating $G \cdot x_i$ and $G \cdot x_j$). The stabilizer of a point in each of these orbits is a maximal parabolic of G, and therefore contains a conjugate of H. It follows that there is a sequence of H-fixed points converging to x, which is a contradiction. ∎

7. Complements

COROLLARY 7.1: *Let M be a closed two manifold admitting an effective action of a simple non-compact Lie group G. Then M is homeomorphic to $T^2, S^2, \mathbb{R}P^2$, or the Klein bottle. The group G is locally isomorphic to $SL(2, \mathbb{R})$, $SL(3, \mathbb{R})$, $SL(2, \mathbb{C})$, or $\mathrm{SO}(2,2)$. All of these groups have almost effective actions on S^2 and $\mathbb{R}P^2$. Only $SL(2, \mathbb{R})$ may act on T^2 and the Klein bottle.*

Proof: By inspecting tables 1 and 3, one finds that G must be locally isomorphic to one of the four groups listed. By theorem 6.9, if $\mathrm{SL}(2, \mathbb{R})$ acts effectively on a closed two manifold M, then $M/\mathrm{SO}(2)$ is homeomorphic to S^1, I, or a point. The last possibility obviously does not occur. By a result of Mostert [10], M is homeomorphic to $T^2, S^2, \mathbb{R}P^2$, or the Klein bottle. Clearly, $\mathrm{SL}(2,\mathbb{R})$ has an effective action on each of these manifolds.

Suppose then that G is one of the groups $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$, or $\mathrm{SO}(2,2)$. Then by theorem 6.6, M is a finite covering of G/P for a maximal parabolic $P \subset G$. One sees by inspection that M is either S^2 or $\mathbb{R}P^2$. ∎

COROLLARY 7.2: *Let M be a closed three manifold admitting an effective action of a simple non-compact Lie group G. Suppose G is not locally isomorphic to $SL(2,\mathbb{R})$. Then M is homeomorphic to one of the following: $S^2 \times S^1$, $\mathbb{R}P^2 \times S^1$, $S^2 \times I/(x,0) \sim (-x,1)$, $\mathbb{R}P^3 \sharp \mathbb{R}P^3$, or S^3/Γ, where Γ is a finite subgroup of $O(3)$. The group G is locally isomorphic to $SL(3,\mathbb{R})$, $SL(4,\mathbb{R})$, $SL(2,\mathbb{C})$,* $\mathrm{SO}(2,3)$, $\mathrm{SU}(2,1)$, or $\mathrm{SO}(4,1)$. *All of the manifolds listed admit an almost effective action of $SL(3,\mathbb{R})$.*

Proof: Referring to tables 1 and 3 we find that G must be locally isomorphic to one of the groups listed. We leave to the reader any subtleties involving finite covers, and assume that G is in fact isomorphic to one of the groups listed. For $G = \mathrm{SL}(4,\mathbb{R})$, $\mathrm{SO}(2,3)$, $\mathrm{SU}(2,1)$, and $\mathrm{SO}(4,1)$ we have $n(G) = 3$, and M is a finite covering of G/S, where S is a maximal parabolic of minimal codimension in G. It is straightforward to show that $\mathrm{SL}(4,\mathbb{R})/S \cong \mathbb{R}P^3$, $\mathrm{SO}(2,3)/S \cong S^2 \times I/(x,0) \sim (-x,1)$, $\mathrm{SO}(4,1)/S \cong \mathbb{R}P^3$, and $\mathrm{SU}(2,1)/S \cong S^3$.

Suppose then that G is isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$. Then by 6.9, M/K is homeomorphic to S^1, I, or a point. In the latter case K acts transitively on M. Since K is isomorphic to $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$, it follows that M is homeomorphic to S^3/Γ, where Γ is a finite subgroup of $O(3)$. We suppose then that M/K is homeomorphic to either S^1 or I. By a result of Mostert [10], M is homeomorphic to $S^2 \times S^1$, $\mathbb{R}P^2 \times S^1$, $S^2 \times I/(x,0) \sim (-x,1)$, $\mathbb{R}P^3 \sharp \mathbb{R}P^3$, S^3, or $\mathbb{R}P^3$. We leave it to the reader to verify that all these manifolds admit an almost effective action of $\mathrm{SL}(3,\mathbb{R})$. ∎

It is well known ([11]) that if a compact group K acts smoothly on a compact manifold M then there is a real representation (ρ, V) of K and a K-equivariant map $\phi : M \to V$. The following corollary gives an analogue for lowest dimensional actions of semisimple groups.

COROLLARY 7.3: *Let G be a semisimple Lie group with finite center. If G acts almost effectively on a closed manifold M of dimension $n(G)$ then there is a representation $\rho : G \to GL(n, \mathbb{R})$, and a G-equivariant, finite-to-one map $\phi : M \to \mathbb{R}P^{n-1}$.*

Proof: By Theorem 6.7, the action of G on M is transitive and G_x is of finite index in $N_G(G_x^0)$. The group $N_G(G_x^0)$ is real algebraic, so by Chevalley's theorem there is a real representation $\rho : G \to GL(n, \mathbb{R})$ such that $\rho(N_G(G_x^0)$ is the stabilizer in $\rho(G)$ of a line L in $\mathbb{R}^n$. The $\rho(G)$ orbit through L is a compact submanifold of $\mathbb{R}P^{n-1}$ diffeomorphic to $\rho(G)/\rho(N_G(G_x^0))$, and the natural map

$$M \cong G/G_x \to \rho(G)/\rho(N_G(G_x^0))$$

is a G-equivariant, finite-to-one embedding of M in $\mathbb{R}P^{n-1}$. ∎

Example 7.4: Let G be a noncompact semisimple Lie group. In this example, we construct a smooth action of G on a closed manifold of dimension $n(G) + 1$ with infinitely many open orbits.

Let f be a smooth real valued function on S^1 with countably infinitely many zeros. The vector field $f\frac{d}{d\theta}$ integrates to give a smooth action of $\mathbb{R}$ on S^1 with infinitely many fixed points. Let P be a maximal parabolic of G, and $\rho : P \to \mathbb{R}$ a nontrivial additive character. We let P act on S^1 via ρ and the action of $\mathbb{R}$ previously defined. Then the induced action of G on $G \times_P S^1$ has infinitely many open orbits. If P is a parabolic of codimension $n(G)$, then $G \times_P S^1$ is a closed manifold of dimension $n(G) + 1$. ∎

Example 7.5: Retaining the notation of the previous example, we construct a continuous action of G on a closed manifold of dimension $n(G) + 1$ with infinitely many open orbits accumulating on a fixed point. Let f be a smooth real valued function on $[-1, 1]$ with countably many zeros accumulating at 1 and suppose also that $f(-1) = 0$. The vector field $f\frac{d}{dt}$ integrates to give a smooth action of $\mathbb{R}$ on the open interval $(-1, 1)$. The manifold $G \times_P (-1, 1)$ has two ends, and the action of G extends continuously to the closed manifold M obtained by taking the one point compactification of each of the ends. The action of G on M has infinitely many open orbits accumulating on a fixed point. ∎

In this paper we have considered only actions on compact manifolds. It is reasonable to ask for the minimum dimension of a smooth manifold, not necessarily compact, on which a connected Lie group G may act. This is equivalent to asking for the minimum codimension of a closed subgroup of G. The topic of maximal subgroups of Lie groups has been studied by several authors (cf. [5], [9]), and in particular it is known [12] that if G is real and semisimple then a maximal subgroup is either parabolic (and therefore cocompact) or reductive (and therefore not cocompact). Since it is known what all the maximal subgroups of real semisimple Lie groups are, it is possible in principle to compute the minimum codimension of a subgroup. It appears to be the case that a subgroup of minimum codimension is always parabolic, although we have not verified this.

Finally, we note that using Theorem 6.7 and results of Mostert one may extend the results of this paper to compact manifolds with boundary, the principle fact being that the boundary components are invariant. We state the relevant theorem and leave the proof to the reader.

THEOREM 7.6: *Let $G = \prod_{i=1}^{k} G_i$ be a connected semisimple Lie group with finite center acting on a compact manifold M of dimension $n(G) + 1$. Then M has at most two boundary components, each of which is of the form G/Q, where Q is a cocompact subgroup of G which does not contain any simple factor of G. Let K be a maximal compact subgroup of G. Then M/K is homeomorphic to S^1, I, or a point. If the boundary of M is nonempty, then M/K is homeomorphic to I.*

8. The tables

In table 1 we have listed the complex simple Lie algebras according to type. In the column labelled "$\mathfrak{s}$" we have listed the maximal parabolics of minimal codimension, the symbol $\mathfrak{s}_\alpha$ denoting the parabolic $\mathfrak{s}(\Pi\backslash\{\alpha\})$ (see §2). The symbol $\mathfrak{h}_{\mathbf{C}}(\mathfrak{g})$ denotes the complex codimension of the maximal parabolic of minimum codimension. The symbol $\Phi(\mathfrak{g})$ denotes the minimum real dimension of a faithful real representation of $\mathfrak{g}$.

In table 2 we have listed the simple real, non-complex Lie algebras, along with their Satake diagrams (cf. Definition 4.4). In the column labelled "$\mathfrak{s}$" we have listed the maximal parabolics of minimal codimension, the symbol $\mathfrak{s}_{\overline{\alpha}}$ denoting the parabolic $\mathfrak{s}(\Pi(\mathfrak{g}, \mathfrak{a})\backslash\{\overline{\alpha}\})$. The symbol $h(\mathfrak{g})$ denotes the real codimension of the maximal parabolic of minimum codimension.

In table 3 we have listed the simple real, non-complex Lie algebras, along with their maximal compact subalgebras (in the column labelled "$\mathfrak{k}$"). In the column labelled "$\Phi(\mathfrak{g})$" we have listed the minimum real dimension of a faithful real representation of $\mathfrak{g}$.

Table 1

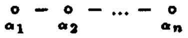
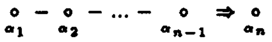
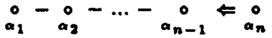
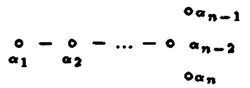
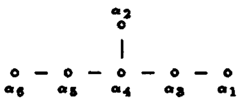
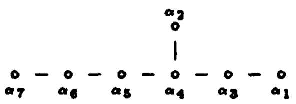
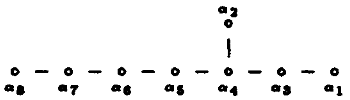
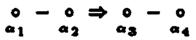
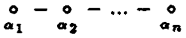
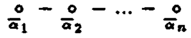
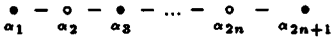
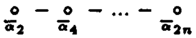
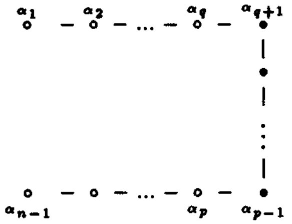
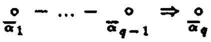
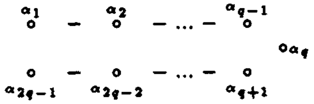
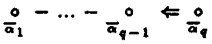
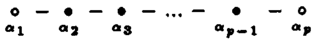
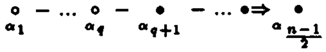
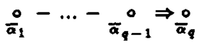
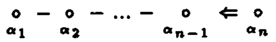
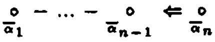
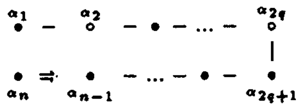
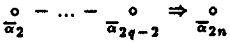
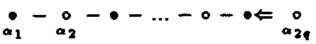
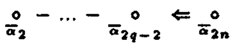
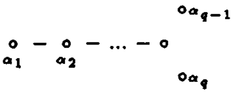
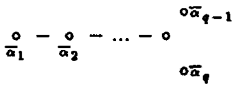
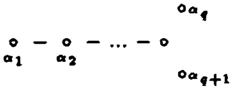
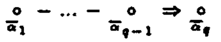
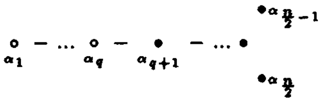
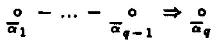
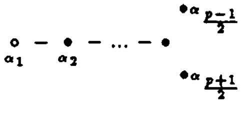
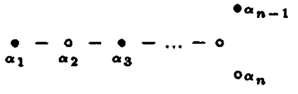
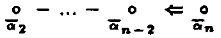
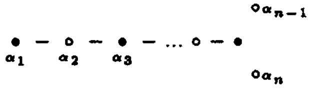
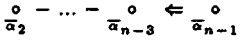
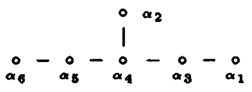
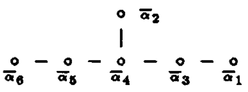
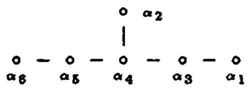
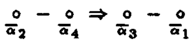
$\mathfrak{g}$	Dynkin diagram	$\mathfrak{s}$	$h_{\mathrm{C}}(\mathfrak{g})$	$\Phi(\mathfrak{g})$
$\mathfrak{a}_n$ $(n\geq 1)$		$\mathfrak{s}_{\alpha_1}, \mathfrak{s}_{\alpha_n}$	n	$2n+2$
$\mathfrak{b}_n$ $(n\geq 2)$		$\mathfrak{s}_{\alpha_1}$ $(n>2)$ $\mathfrak{s}_{\alpha_2}$ $(n=2)$	$2n-1$	$4n+2$ 8
$\mathfrak{c}_n$ $(n\geq 3)$		$\mathfrak{s}_{\alpha_1}$	$2n-1$	$4n$
$\mathfrak{d}_n$ $(n\geq 4)$		$\mathfrak{s}_{\alpha_1}$	$2n-2$	$4n$
$\mathfrak{e}_6$		$\mathfrak{s}_{\alpha_1}, \mathfrak{s}_{\alpha_6}$	14	54
$\mathfrak{e}_7$		$\mathfrak{s}_{\alpha_7}$	27	112
$\mathfrak{e}_8$		$\mathfrak{s}_{\alpha_8}$	57	496
$\mathfrak{f}_4$		$\mathfrak{s}_{\alpha_1}, \mathfrak{s}_{\alpha_4}$	15	52
$\mathfrak{g}_2$		$\mathfrak{s}_{\alpha_1}, \mathfrak{s}_{\alpha_2}$	5	14

Table 2

$\mathfrak{g}$	Satake diagram	Dynkin diagram	$\mathfrak{s}$	$h(\mathfrak{g})$
$\mathfrak{sl}(n+1,\mathbf{R})$			$\mathfrak{s}_{\overline{\alpha}_1}, \mathfrak{s}_{\overline{\alpha}_n}$	n
$\mathfrak{sl}(n+1,\mathbf{H})$			$\mathfrak{s}_{\overline{\alpha}_2}, \mathfrak{s}_{\overline{\alpha}_{2n}}$	$4n$
$\mathfrak{su}(p,q)$ $p>q>1$			$\mathfrak{s}_{\overline{\alpha}_1}$	$2(p+q)-3$
$\mathfrak{su}(q,q)$ $(q>1)$			$\mathfrak{s}_{\overline{\alpha}_1}$ $(q>2)$ $\mathfrak{s}_{\overline{\alpha}_2}$ $(q=2)$	$4q-3$ 4
$\mathfrak{su}(p,1)$			$\mathfrak{s}_{\overline{\alpha}_1}$	$2p-1$
$\mathfrak{so}(p,q)$ $p\geq q>1$ $p+q$ odd			$\mathfrak{s}_{\overline{\alpha}_1}$	$p+q-2$
$\mathfrak{so}(p,1)$ p even			$\mathfrak{s}_{\overline{\alpha}_1}$	$p-1$
$\mathfrak{sp}(n,\mathbf{R})$			$\mathfrak{s}_{\overline{\alpha}_1}$	$2n-1$
$\mathfrak{sp}(p,q)$ $p>q$			$\mathfrak{s}_{\overline{\alpha}_2}$	$4(p+q)-5$

$\mathfrak{g}$	Satake diagram	Dynkin diagram	$\mathfrak{s}$	$h(\mathfrak{g})$
$\mathfrak{sp}(q,q)$ $q>1$			$\mathfrak{s}_{\bar{\alpha}_2}$ $(q>2)$ $\mathfrak{s}_{\bar{\alpha}_4}$ $(q=2)$	$8q-5$ 10
$\mathfrak{so}(q,q)$ $q\geq 4$			$\mathfrak{s}_{\bar{\alpha}_1}$	$2q-2$
$\mathfrak{so}(q+2,q)$ $q>1$			$\mathfrak{s}_{\bar{\alpha}_1}$	$2q$
$\mathfrak{so}(p,q)$ $p-2>q>1$ $p+q$ even			$\mathfrak{s}_{\bar{\alpha}_1}$	$p+q-2$
$\mathfrak{so}(p,1)$ $p>1$ p odd			$\mathfrak{s}_{\bar{\alpha}_1}$	$p-1$
$\mathfrak{so}^*(2n)$ $n>3$ n even			$\mathfrak{s}_{\bar{\alpha}_2}$	$4n-7$
$\mathfrak{so}^*(2n)$ $n>3$ n odd			$\mathfrak{s}_{\bar{\alpha}_2}$	$4n-7$
$\mathfrak{e}_{6(6)}$			$\mathfrak{s}_{\bar{\alpha}_1},\mathfrak{s}_{\bar{\alpha}_6}$	14
$\mathfrak{e}_{6(2)}$			$\mathfrak{s}_{\bar{\alpha}_2}$	21

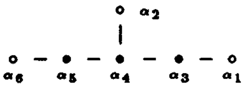
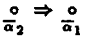
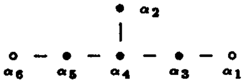
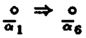
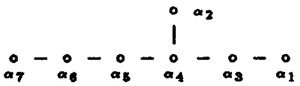
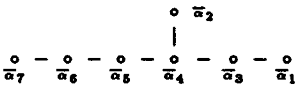
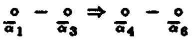
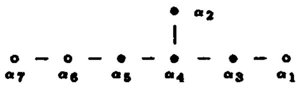
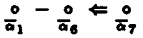
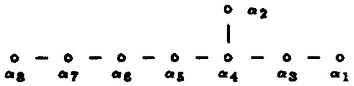
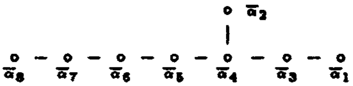
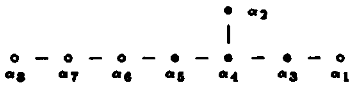
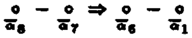
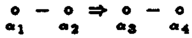
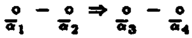
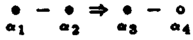
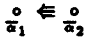
$\mathfrak{g}$	Satake diagram	Dynkin diagram	$\mathfrak{s}$	$h(\mathfrak{g})$
$\mathfrak{e}_{6(14)}$			$\mathfrak{s}_{\overline{\alpha}_2}$	21
$\mathfrak{e}_{6(-26)}$			$\mathfrak{s}_{\overline{\alpha}_1}, \mathfrak{s}_{\overline{\alpha}_6}$	14
$\mathfrak{e}_{7(7)}$			$\mathfrak{s}_{\overline{\alpha}_7}$	27
$\mathfrak{e}_{7(-5)}$			$\mathfrak{s}_{\overline{\alpha}_1}$	33
$\mathfrak{e}_{7(-25)}$			$\mathfrak{s}_{\overline{\alpha}_7}$	27
$\mathfrak{e}_{8(8)}$			$\mathfrak{s}_{\overline{\alpha}_8}$	57
$\mathfrak{e}_{8(-24)}$			$\mathfrak{s}_{\overline{\alpha}_8}$	57
$\mathfrak{f}_{4(4)}$			$\mathfrak{s}_{\overline{\alpha}_1}, \mathfrak{s}_{\overline{\alpha}_4}$	15
$\mathfrak{f}_{4(-20)}$			$\mathfrak{s}_{\overline{\alpha}_4}$	15
$\mathfrak{g}_{2(2)}$			$\mathfrak{s}_{\overline{\alpha}_1}, \mathfrak{s}_{\overline{\alpha}_2}$	5

Table 3

Type	$\mathfrak{g}$	$\mathfrak{k}$	$h(\mathfrak{g})$	$\Phi(\mathfrak{g})$
AI	$\mathfrak{sl}(n, \mathbb{R})$	$\mathfrak{so}(n)$	$n-1$	n
AII	$\mathfrak{sl}(n, \mathbb{H})$	$\mathfrak{sp}(n)$	$4n-4$	$4n$ $(n>2)$ 6 $(n=2)$
AIII	$\mathfrak{su}(p, q)$ $p\geq q\geq 1$	$\mathfrak{s}(\mathfrak{u}(p)\times\mathfrak{u}(q))$	$2(p+q)-3$ $((p,q)\neq(2,2))$ 4 $(p=q=2)$	$4(p+q)$ $(p+q>4)$ 6 $(p+q=3,4)$
BDI	$\mathfrak{so}(p, q)$ $(p+q>4)$	$\mathfrak{so}(p)\times\mathfrak{so}(q)$	$p+q-2$ $((p,q)\neq(3,3))$ 3 $(p=q=3)$	$p+q$
DIII	$\mathfrak{so}^*(2n)$	$\mathfrak{u}(n)$	$4n-7$ $(n>4)$ 3 $(n=2)$ 6 $(n=4)$	$4n$ $(n>4)$ 6 $(n=2)$ 8 $(n=4)$
CI	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{u}(n)$	$2n-1$	$2n$
CII	$\mathfrak{sp}(p, q)$ $p+q>3$	$\mathfrak{sp}(p)\times\mathfrak{sp}(q)$	$4(p+q)-5$ $((p,q)\neq(2,2))$ 10 $(p=q=2)$	$4(p+q)$
EI	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4)$	14	≥ 27
EII	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(6)\times\mathfrak{su}(2)$	21	≥ 27

Type	$\mathfrak{g}$	$\mathfrak{k}$	$h(\mathfrak{g})$	$\Phi(\mathfrak{g})$
EIII	$\mathfrak{e}_{6(14)}$	$\mathfrak{so}(10) \times \mathfrak{so}(2)$	21	≥ 27
EIV	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-52)}$	14	≥ 27
EV	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(8)$	27	≥ 56
EVI	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(12) \times \mathfrak{su}(2)$	33	≥ 56
EVII	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-78)} \times \mathfrak{so}(2)$	27	≥ 56
EVIII	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$	57	≥ 248
EIX	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-133)} \times \mathfrak{su}(2)$	57	≥ 248
FI	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3) \times \mathfrak{su}(2)$	15	≥ 26
FII	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$	15	≥ 26
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2) \times \mathfrak{su}(2)$	5	≥ 7

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